239 Open Mathematical Olympiad 2013

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– Grade 10-11

1 Consider all permutations of natural numbers from 1 to 100. A permutation is called *double* when it has the following property: If you write this permutation twice in a row, then delete 100 numbers from them you get the remaining numbers $1, 2, 3, \ldots, 100$ in order. How many *double* permutations are there?

2 For some 99-digit number k, there exist two different 100-digit numbers n such that the sum of all natural numbers from 1 to n ends in the same 100 digits as the number kn, but is not equal to it. Prove that $k - 3$ is divisible by 5.

3 Inside a regular triangle ABC, points X and Y are chosen such that $\angle AXC = 120^\circ$, $2\angle XAC + \angle YBC = 90^\circ$and $XY = YB = \frac{AC}{\sqrt{3}}$. Prove that point Y lies on the incircle of triangle ABC.

4 For positive numbers a, b, c satisfying condition $a + b + c < 2$, Prove that

$$\sqrt{a^2 + bc} + \sqrt{b^2 + ca} + \sqrt{c^2 + ab} < 3.$$

5 A squirrel has infinitely many nuts; one nut of each of the masses $1g, 2g, 3g, \ldots$. The squirrel took 100 bags, in each put a finite number of nuts, after which wrote on each bag the total mass of the nuts inside it. Prove that it is possible to create bags of the same mass using no more than 500 nuts.

6 Convex polyhedron M with triangular faces is cut into tetrahedrons; all the vertices of the tetrahedrons are the vertices of the polyhedron, and any two tetrahedrons either do not intersect, or they intersect along a common vertex, common edge, or common face. Prove that it it's not possible that each tetrahedron has exactly one face on the surface of M.

7 Point M is the midpoint of side BC of convex quadrilateral $ABCD$. If $\angle AMD < 120^\circ$. Prove that

$$(AB + AM)^2 + (CD + DM)^2 > AD \cdot BC + 2AB \cdot CD.$$

8 Prove that if you choose 10^{100} points on a circle and arrange numbers from 1 to 10^{100} on them in some order, then you can choose 100 pairwise disjoint chords with ends at the selected points such that the sums of the numbers at the ends of all of them are equal to each other.

–	Grade 8-9
1	Among the divisors of a natural number n, we have numbers such that when they are devided by 2013, give us remainders $1001, 1002, \dots, 2012$. Prove that among the divisors of the number n^2, there exist numbers such that when they are divided by 2013, give us reminders $1, 2, 3, \dots, 2012$.
2	In the set A with n elements, $[\sqrt{2n}] + 2$ subsets are chosen such that the union of any three of them is equal to A. Prove that the union of any two of them is equal to A as well.
3	The altitudes AA_1 and CC_1 of an acute-angled triangle ABC intersect at point H. A straight line passing through H parallel to line A_1C_1 intersects the circumscribed circles of triangles AHC_1 and CHA_1 at points X and Y, respectively. Prove that points X and Y are equidistant from the midpoint of segment BH.
4	We are given a graph G with n edges. For each edge, we write down the lesser degree of two vertices at the end of that edge. Prove that the sum of the resulting n numbers is at most $100n\sqrt{n}$.
5	Same as grade 10-11, 3.
6	A quarter of an checkered plane is given, infinite to the right and up. All its rows and columns are numbered starting from 0. All cells with coordinates $(2n, n)$, were cut out from this figure, starting from $n = 1$. In each of the remaining cells they wrote a number, the number of paths from the corner cell to this one (you can only walk up and to the right and you cannot pass through the removed cells). Prove that for each removed cell the numbers to the left and below it differ by exactly 2.
7	Dima wrote several natural numbers on the blackboard and underlined some of them. Misha wants to erase several numbers (but not all) such that a multiple of three underlined numbers remain and the total amount of the remaining numbers would be divisible by 2013; but after trying for a while he realizes that it's impossible to do this. What is the largest number of the numbers on the board?
8	The product of the positive numbers a, b, c, d and e is equal to 1. Prove that $$\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{d^2}+\frac{d^2}{e^2}+\frac{e^2}{a^2} \geq a+b+c+d+e.$$
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