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P1 The function $f(x) = x^2 + \sin x$ and the sequence of positive numbers $\{a_n\}$ satisfy $a_1 = 1$, $f(a_n) = a_{n-1}$, where $n \geq 2$. Prove that there exists a positive integer n such that $a_1 + a_2 + \cdots + a_n > 2020$.

P2 In $\triangle ABC$, $AB > AC$. Let O and I be the circumcenter and incenter respectively. Prove that if $\angle AIO = 30^\circ$, then $\angle ABC = 60^\circ$.

P3 A set of k integers is said to be a *complete residue system modulo* k if no two of its elements are congruent modulo k. Find all positive integers m so that there are infinitely many positive integers n wherein $\{1^n, 2^n, \dots, m^n\}$ is a complete residue system modulo m.

P4 Two students A and B play a game on a 20 x 20 chessboard. It is known that two squares are said to be *adjacent* if the two squares have a common side. At the beginning, there is a chess piece in a certain square of the chessboard. Given that A will be the first one to move the chess piece, A and B will alternately move this chess piece to an adjacent square. Also, the common side of any pair of adjacent squares can only be passed once. If the opponent cannot move anymore, then he will be declared the winner (to clarify since the wording wasnt that good, you lose if you cant move). Who among A and B has a winning strategy? Justify your claim.

P5 Find all positive integers a so that for any $\left\lfloor \frac{a+1}{2} \right\rfloor$-digit number that is composed of only digits 0 and 2 (where 0 cannot be the first digit) is not a multiple of a.

CNMO Basic Level

BP1 For all positive real numbers a, b, c, prove that

$$\frac{a^3+b^3}{\sqrt{a^2-ab+b^2}} + \frac{b^3+c^3}{\sqrt{b^2-bc+c^2}} + \frac{c^3+a^3}{\sqrt{c^2-ca+a^2}} \geq 2(a^2+b^2+c^2)$$

BP2 Given $a, b, c \in \mathbb{R}$ satisfying $a + b + c = a^2 + b^2 + c^2 = 1$, show that $\frac{-1}{4} \leq ab \leq \frac{4}{9}$.

BP3 Are there infinitely many positive integers n such that $19|1 + 2^n + 3^n + 4^n$? Justify your claim.

BP4 In $\triangle ABC$, $\angle BAC = 60^\circ$, point D lies on side BC, O_1 and O_2 are the centers of the circumcircles of $\triangle ABD$ and $\triangle ACD$, respectively. Lines BO_1 and CO_2 intersect at point P. If I is the incenter of $\triangle ABC$ and H is the orthocenter of $\triangle PBC$, then prove that the four points B, C, I, H are on the same circle.

BP5 It is known that subsets $A_1, A_2, \cdots, A_n$ of set $I = \{1, 2, \cdots, 101\}$ satisfy the following condition

For any i, j $(1 \le i < j \le n)$, there exists $a, b \in A_i \cap A_j$ so that $(a, b) = 1$

Determine the maximum positive integer n.

*(a, b) means $\gcd(a, b)$