South East Mathematical Olympiad 2020

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–	Grade 10
–	Day 1
1	Let $f(x)=a(3a+2c)x^2-2b(2a+c)x+b^2+(c+a)^2$ $(a,b,c\in R, a(3a+2c)\neq 0)$. If $$f(x)\leq 1$$ for any real x, find the maximum of $\lvert ab\rvert$.
2	In a scalene triangle ΔABC, $AB<AC$, PB and PC are tangents of the circumcircle (O) of ΔABC. A point R lies on the arc $\widehat{AC}$(not containing B), PR intersects (O) again at Q. Suppose I is the incenter of ΔABC, $ID\perp BC$ at D, QD intersects (O) again at G. A line passing through I and perpendicular to AI intersects AB, AC at M, N, respectively. Prove that, if $AR \parallel BC$, then A, G, M, N are concyclic.
3	Given a polynomial $f(x)=x^{2020}+\sum_{i=0}^{2019}c_ix^i$, where $c_i\in\{-1,0,1\}$. Denote N the number of positive integer roots of $f(x)=0$ (counting multiplicity). If $f(x)=0$ has no negative integer roots, find the maximum of N.
4	Let $a_1,a_2,\ldots,a_{17}$ be a permutation of $1,2,\ldots,17$ such that $(a_1-a_2)(a_2-a_3)\ldots(a_{17}-a_1)=n^{17}$ .Find the maximum possible value of n .
–	Day 2
5	Consider the set $I=\{1,2,\cdots,2020\}$. Let $W=\{w(a,b)=(a+b)+ab\vert a,b\in I\}\cap I$, $Y=\{y(a,b)=(a+b)\cdot ab\vert a,b\in I\}\cap I$ be its two subsets. Further, let $X=W\cap Y$. **(1)** Find the sum of maximal and minimal elements in X. **(2)** An element $n=y(a,b)(a\leq b)$ in Y is called *excellent*, if its representation is not unique (for instance, $20=y(1,5)=y(2,3)$). Find the number of *excellent* elements in Y. **(2)** is only for Grade 11.
6	In a quadrilateral $ABCD$, $\angle ABC=\angle ADC<90^\circ$. The circle with diameter AC intersects BC and CD again at E, F, respectively. M is the midpoint of BD, and $AN\perp BD$ at N. Prove that M, N, E, F is concyclic.

7 Given any prime $p \geq 3$. Show that for all sufficient large positive integer x, at least one of $x+1, x+2, \cdots, x+\frac{p+3}{2}$ has a prime divisor greater than p.

8 Using a nozzle to paint each square in a $1 \times n$ stripe, when the nozzle is aiming at the i-th square, the square is painted black, and simultaneously, its left and right neighboring square (if exists) each has an independent probability of $\frac{1}{2}$ to be painted black.

In the optimal strategy (i.e. achieving least possible number of painting), the expectation of number of painting to paint all the squares black, is $T(n)$. Find the explicit formula of $T(n)$.

– Grade 11

– Day 1

1 Let $a_1, a_2, \ldots, a_{17}$ be a permutation of $1, 2, \ldots, 17$ such that $(a_1 - a_2)(a_2 - a_3) \ldots (a_{17} - a_1) = 2^n$. Find the maximum possible value of positive integer n .

2 In a scalene triangle ΔABC, $AB < AC$, PB and PC are tangents of the circumcircle (O) of ΔABC. A point R lies on the arc $\widehat{AC}$(not containing B), PR intersects (O) again at Q. Suppose I is the incenter of ΔABC, $ID \perp BC$ at D, QD intersects (O) again at G. A line passing through I and perpendicular to AI intersects AG, AC at M, N, respectively. S is the midpoint of arc $\widehat{AR}$, andSN intersects (O) again at T.
Prove that, if $AR \parallel BC$, then M, B, T are collinear.

3 Same as Grade 10 P3

4 Let $0 \leq a_1 \leq a_2 \leq \ldots \leq a_{n-1} \leq a_n$ such that $a_1 + a_2 + \ldots + a_n = 1$. Prove for any non-negative numbers $x_1, x_2, \ldots, x_n$, $y_1, y_2, \ldots, y_n$ the inequality

$$\left(\sum_{i=1}^{n} a_i x_i - \prod_{i=1}^{n} x_i^{a_i}\right)\left(\sum_{i=1}^{n} a_i y_i - \prod_{i=1}^{n} y_i^{a_i}\right) \leq a_n^2 \left(n\sqrt{\sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i} - \sum_{i=1}^{n}\sqrt{x_i}\sum_{i=1}^{n}\sqrt{y_i}\right)^2 .$$

– Day 2

5 Same as Grade 10 P5 (with part 2.)

6 Same as Grade 10 P6

7 Arrange all square-free positive integers in ascending order $a_1, a_2, a_3, \ldots, a_n, \ldots$. Prove that there are infinitely many positive integers n, such that $a_{n+1} - a_n = 2020$.

8 Same as Grade 10 P8.