China Girls Math Olympiad 2020

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Day 1 August 9, 2020

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1 In the quadrilateral $ABCD$, $AB=AD$, $CB=CD$, $\angle ABC=90^\circ$. E, F are on AB, AD and P, Q are on EF(P is between E,Q), satisfy $\frac{AE}{EP}=\frac{AF}{FQ}$. X,Y are on CP,CQ that satisfy $BX\perp CP, DY\perp CQ$. Prove that X,P,Q,Y are concyclic.

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2 Let n be an integer and $n\geq 2$, $x_1,x_2,\cdots,x_n$ are arbitrary real number, find the maximum value of

$$2\sum_{1\leq i<j\leq n}\lfloor x_ix_j\rfloor-(n-1)\sum_{i=1}^n\lfloor x_i^2\rfloor$$

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3 There are 3 classes with n students in each class, and the heights of all $3n$ students are pairwise distinct. Partition the students into groups of 3 such that in each group, there is one student from each class. In each group, call the tallest student the *tall guy*. Suppose that for any partition of the students, there are at least 10 tall guys in each class, prove that the minimum value of n is 40.

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4 Let p,q be primes, where $p>q$. Define $t=\gcd(p!-1,q!-1)$. Prove that $t\leq p^{\frac{p}{3}}$.

Day 2 August 10, 2020

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5 Find all the real number sequences $\{b_n\}_{n \geq 1}$ and $\{c_n\}_{n \geq 1}$ that satisfy the following conditions:
(i) For any positive integer n, $b_n \leq c_n$;
(ii) For any positive integer n, b_{n+1} and c_{n+1} is the two roots of the equation $x^2+b_nx+c_n=0$.

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6 Let p,q be integers and $p,q>1$, $gcd(p,\,6q)=1$. Prove that:

$$\sum_{k=1}^{q-1}\left\lfloor\frac{pk}{q}\right\rfloor^2\equiv 2p\sum_{k=1}^{q-1}k\left\lfloor\frac{pk}{q}\right\rfloor\,(mod\,q-1)$$

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7 Let O be the circumcenter of triangle $\triangle ABC$, where $\angle BAC=120^\circ$. The tangent at A to (ABC) meets the tangents at B,C at (ABC) at points P,Q respectively. Let H,I be the orthocenter and incenter of $\triangle OPQ$ respectively. Define M,N as the midpoints of arc BAC and OI respectively, and let MN meet (ABC) again at D. Prove that AD is perpendicular to HI.

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8 Let n be a given positive integer. Let $\mathbb{N}_+$ denote the set of all positive integers.

Determine the number of all finite lists $(a_1, a_2, \cdots, a_m)$ such that:
(1) $m \in \mathbb{N}_+$ and $a_1, a_2, \cdots, a_m \in \mathbb{N}_+$ and $a_1 + a_2 + \cdots + a_m = n$.
(2) The number of all pairs of integers (i, j) satisfying $1 \le i < j \le m$ and $a_i > a_j$ is even.

For example, when $n = 4$, the number of all such lists $(a_1, a_2, \cdots, a_m)$ is 6, and these lists are (4), $(1, 3)$, $(2, 2)$, $(1, 1, 2)$, $(2, 1, 1)$, $(1, 1, 1, 1)$.