Math olympiad for the French Speaking, 2020

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by parmenides51, anyone__42

–	Juniors
1	Let ABC be a triangle such that $AB < AC$, ω its inscribed circle and Γ its circumscribed circle. Let also ω_b be the excircle relative to vertex B, then B' is the point of tangency between ω_b and (AC). Similarly, let the circle ω_c be the excircle exinscribed relative to vertex C, then C' is the point of tangency between ω_c and (AB). Finally, let I be the center of ω and X the point of Γ such that $\angle XAI$ is a right angle. Prove that the triangles XBC' and XCB' are congruent.
2	Emperor Zorg wishes to found a colony on a new planet. Each of the n cities that he will establish there will have to speak exactly one of the Empire's 2020 official languages. Some towns in the colony will be connected by a direct air link, each link can be taken in both directions. The emperor fixed the cost of the ticket for each connection to 1 galactic credit. He wishes that, given any two cities speaking the same language, it is always possible to travel from one to the other via these air links, and that the cheapest trip between these two cities costs exactly 2020 galactic credits. For what values of n can Emperor Zorg fulfill his dream?
3	Let n be an integer greater than or equal to 1. Find, as a function of n, the smallest integer $k \geq 2$ such that, among any k real numbers, there are necessarily two of which the difference, in absolute value, is either strictly less than $1/n$, either strictly greater than n.
4	Find all the integers x, y and z greater than or equal to 0 such that $2^x + 9 \cdot 7^y = z^3$
–	Seniors
1	Let ABC be an acute triangle with $AC > AB$, Let DEF be the intouch triangle with $D \in (BC)$,$E \in (AC)$,$F \in (AB)$,, let G be the intersecttion of the perpendicular from D to EF with AB, and $X = (ABC) \cap (AEF)$. Prove that B, D, G and X are concylic
2	Let $a_1, a_2, \ldots, a_n$ be a finite sequence of non negative integers, its subsequences are the sequences of the form $a_i, a_{i+1}, \ldots, a_j$ with $1 \leq i \leq j \leq n$. Two subsequences are said to be equal if they have the same length and have the same terms, that is, two subsequences $a_i, a_{i+1}, \ldots, a_j$ and $a_u, a_{u+1}, \ldots a_v$ are equal iff $j - i = u - v$ and $a_{i+k} = a_{u+k}$ forall integers k such that $0 \leq k \leq j - 1$. Finally, we say that a subsequence $a_i, a_{i+1}, \ldots, a_j$ is palindromic if $a_{i+k} = a_{j-k}$ forall integers k such that $0 \leq k \leq j - i$ What is the greatest number of different palindromic subsequences that can a palindromic sequence of length n contain?

3 Let $(a_i)_{i\in\mathbb{N}}$ be a sequence with $a_1 = \frac{3}{2}$ such that

$$a_{n+1} = 1 + \frac{n}{a_n}$$

Find n such that $2020 \leq a_n < 2021$

4 Let $(a_i)_{i\in\mathbb{N}}$ a sequence of positive integers, such that for any finite, non-empty subset S of $\mathbb{N}$, the integer

$$\Pi_{k\in S} a_k - 1$$

is prime.
Prove that the number of a_i's with $i \in \mathbb{N}$ such that a_i has less than m distincts prime factors is finite.