Problems from Year 30 USAMTS (2018-2019)

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– Round 1 (due 10/15/18)

1: Fill in each space of the grid with one of the numbers $1, 2, \ldots, 30$, using each number once. For $1 \leq n \leq 29$, the two spaces containing n and $n+1$ must be in either the same row or the same column. Some numbers have been given to you.

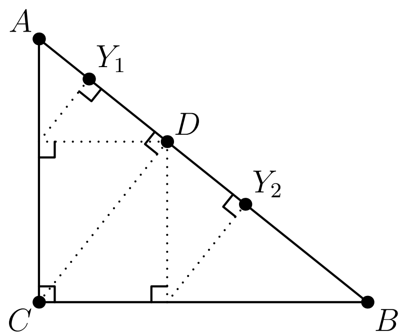
29					
	19			17	
13			21		8
	4		15		24
10				26	

You do not need to prove that your answer is the only one possible; you merely need to find an answer that satisfies the constraints above. (Note: In any other USAMTS problem, you need to provide a full proof. Only in this problem is an answer without justification acceptable.)

2: Let $n > 1$ be an integer. There are n orangutoads, conveniently numbered $1, 2, \ldots, n$, each sitting at an integer position on the number line. They take turns moving in the order $1, 2, \ldots, n$, and then going back to 1 to start the process over; they stop if any orangutoad is ever unable to move. To move, an orangutoad chooses another orangutoad who is at least 2 units away from her towards them by a a distance of 1 unit. (Multiple orangutoads can be at the same position.) Show that eventually some orangutoad will be unable to move.

3: Find, with proof, all pairs of positive integers (n, d) with the following property: for every integer S, there exists a unique non-decreasing sequence of n integers $a_1, a_2, ..., a_n$ such that $a_1 + a_2 + ... + a_n = S$ and $a_n - a_1 = d$.

4: Right triangle $\triangle ABC$ has $\angle C = 90°$. A fly is trapped inside $\triangle ABC$. It starts at point D, the foot of the altitude from C to $\overline{AB}$, and then makes a (finite) sequence of moves. In each move, it flies in a direction parallel to either $\overline{AC}$ or $\overline{BC}$; upon reaching a leg of the triangle, it then flies to a point on $\overline{AB}$ in a direction parallel to $\overline{CD}$. For example, on its first move, the fly can move to either of the points Y_1 or Y_2, as shown.



Let P and Q be distinct points on $\overline{AB}$. Show that the fly can reach some point on $\overline{PQ}$.

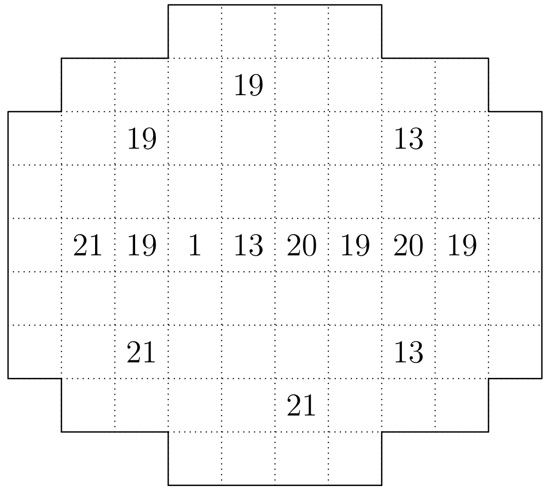
5: A positive integer is called *uphill* if the digits in its decimal representation form a non-decreasing sequence from left to right. That is, a number with decimal representation $\overline{a_1a_2\cdots a_d}$ is uphill if $a_i \le a_{i+1}$ for all i
(All single-digit integers are uphill.)

Given a positive integer n, let $f(n)$ be the smallest nonnegative integer m such that $n+m$ is uphill. For example, $f(520)=35$ and $f(169)=0$. Find, with proof, the value of

$$f(1)-f(2)+f(3)-f(4)+\cdots+f(10^{2018}-1)$$

– Round 2 (due 11/26/18)

1: The grid to the right consists of 74 unit squares, marked by gridlines. Partition the grid into five regions along the gridlines so that the areas of the regions are 1, 13, 19, 20, and 21. A square with a number should be contained in the region with that area.



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2: Given a set of positive integers R, we define the *friend set* of R to be all positive integers that are divisible by at least one number in R. The friend set of R is denoted by $\mathcal{F}(S_1) = \mathcal{F}(S_2)$. Show that $S_1 = S_2$.

3: Alice, Bob, and Chebyshev play a game. Alice puts six red chips into a bag, Bob puts seven blue chips into the bag, and Chebyshev puts eight green chips into the bag. Then, the almighty Zan randomly removes chips from the bag one at a time and gives them back to the corresponding player. The winner of the game is the first player to get all of their chips back. Find, with proof, the probability that Bob wins the game.

4: Find, with proof, all ordered pairs of positive integers (a, b) with the following property: there exist positive integers r, s, and t such that for all n for which both sides are defined,

$$\binom{\binom{n}{a}}{b} = r\binom{n+s}{t}.$$

5: Acute scalene triangle $\triangle ABC$ has circumcenter O and orthocenter H. Points X and Y, distinct from B and C, lie on the circumcircle of $\triangle ABC$ such that $\angle BXH = \angle CYH = 90^\circ$. Show that if lines XY, AH, and BC are concurrent, then OH is parallel to BC.

– Round 3 (due 1/2/19)

1: Fill in each white hexagon with a positive digit from 1 to 9. Some digits have been given to you.

Each of the seven gray hexagons touches six hexagons; these six hexagons must contain six distinct digits, and the sum of these six digits must equal the number inside the gray hexagon.
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You do not need to prove that your answer is the only one possible; you merely need to find an answer that satisfies the constraints above. (Note: In any other USAMTS problem, you need to provide a full proof. Only in this problem is an answer without justification acceptable.)

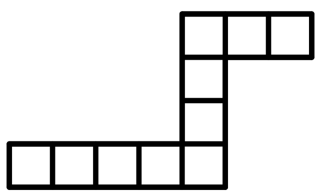
2: Lizzie writes a list of fractions as follows. First, she writes $\frac{1}{1}$, the only fraction whose numerator and denominator add to 2. Then she writes the two fractions whose numerator and denominator add to 3, in increasing order of denominator. Then she writes the three fractions whose numerator and denominator sum to 4 in increasing order of denominator. She continues in this way until she has written all the fractions whose numerator and denominator sum to at most 1000. So Lizzie's list looks like:

$$\frac{1}{1}, \frac{2}{1}, \frac{1}{2}, \frac{3}{1}, \frac{2}{2}, \frac{1}{3}, \frac{4}{1}, \frac{3}{2}, \frac{2}{3}, \frac{1}{4}, \ldots, \frac{1}{999}.$$

Let p_k be the product of the first k fractions in Lizzie's list. Find, with proof, the value of $p_1 + p_2 + \ldots + p_{499500}$.

3: Cyclic quadrilateral $ABCD$ has $AC \perp BD$, $AB+CD = 12$, and $BC+AD = 13$. FInd the greatest possible area of $ABCD$.

4: An eel is a polyomino formed by a path of unit squares which makes two turns in opposite directions (note that this means the smallest eel has four cells). For example, the polyomino shown below is an eel. What is the maximum area of a 1000×1000 grid of unit squares that can be covered by eels without overlap?



5: The sequence $\{a_n\}$ is defined by $a_0 = 1, a_1 = 2$, and for $n \geq 2$,

$$a_n = a_{n-1}^2 + (a_0 a_1 \ldots a_{n-2})^2.$$

Let k be a positive integer, and let p be a prime factor of a_k. Show that $p > 4(k-1)$.

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