Harvard-MIT Mathematics Tournament 2018

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– Algebra and Number Theory

1 For some real number c, the graphs of the equation $y = |x-20| + |x+18|$ and the line $y = x + c$ intersect at exactly one point. What is c?

2 Compute the positive real number x satisfying

$$x^{(2x^6)} = 3.$$

3 There are two prime numbers p so that $5p$ can be expressed in the form $\left\lfloor \frac{n^2}{5} \right\rfloor$ for some positive integer n. What is the sum of these two prime numbers?

4 Distinct prime numbers p, q, r satisfy the equation

$$2pqr + 50pq = 7pqr + 55pr = 8pqr + 12qr = A$$

for some positive integer A. What is A?

5 Let $\{\omega_1, \omega_2, \cdots, \omega_{100}\}$ be the roots of $\frac{x^{101}-1}{x-1}$ (in some order). Consider the set

$$S = \{\omega_1^1, \omega_2^2, \omega_3^3, \cdots, \omega_{100}^{100}\}.$$

Let M be the maximum possible number of unique values in S, and let N be the minimum possible number of unique values in S. Find $M - N$.

6 Let α, β, and γ be three real numbers. Suppose that

$$\cos\alpha + \cos\beta + \cos\gamma = 1$$

$$\sin\alpha + \sin\beta + \sin\gamma = 1.$$

Find the smallest possible value of $\cos\alpha$.

7 Rachel has the number 1000 in her hands. When she puts the number x in her left pocket, the number changes to $x+1$. When she puts the number x in her right pocket, the number changes to x^{-1}. Each minute, she flips a fair coin. If it lands heads, she puts the number into her left pocket, and if it lands tails, she puts it into her right pocket. She then takes the new number out of her pocket. If the expected value of the number in Rachel's hands after eight minutes is E, compute $\left\lfloor \frac{E}{10} \right\rfloor$.

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8 For how many pairs of sequences of nonnegative integers $(b_1, b_2, \dots, b_{2018})$ and $(c_1, c_2, \dots, c_{2018})$ does there exist a sequence of nonnegative integers $(a_0, \dots, a_{2018})$ with the following properties:

- For $0 \le i \le 2018, a_i < 2^{2018}$.
- For $1 \le i \le 2018, b_i = a_{i-1} + a_i$ and $c_i = a_{i-1} | a_i$;

where $|$ denotes the bitwise or operation?

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9 Assume the quartic $x^4 - ax^3 + bx^2 - ax + d = 0$ has four real roots $\frac{1}{2} \le x_1, x_2, x_3, x_4 \le 2$. Find the maximum possible value of $\frac{(x_1+x_2)(x_1+x_3)x_4}{(x_4+x_2)(x_4+x_3)x_1}$.

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10 Let S be a randomly chosen 6-element subset of the set $\{0, 1, 2, \dots, n\}$. Consider the polynomial $P(x) = \sum_{i \in S} x^i$. Let X_n be the probability that $P(x)$ is divisible by some nonconstant polynomial $Q(x)$ of degree at most 3 with integer coefficients satisfying $Q(0) \neq 0$. Find the limit of X_n as n goes to infinity.

– Combinatorics

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1 Consider a 2×3 grid where each entry is either 0, 1, or 2. For how many such grids is the sum of the numbers in every row and in every column a multiple of 3? One valid grid is shown below:

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \end{bmatrix}$$

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2 Let $a < b$ be five-digit palindromes (without leading zeroes) such that there is no other five-digit palindrome strictly between a and b. What are all possible values of $b - a$? (A number is a palindrome if it reads the same forwards and backwards in base 10.)

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3 A 4×4 window is made out of 16 square windowpanes. How many ways are there to stain each of the windowpanes, red, pink, or magenta, such that each windowpane is the same color as exactly two of its neighbors? Two different windowpanes are neighbors if they share a side.

4 How many ways are there for Nick to travel from $(0,0)$ to $(16,16)$ in the coordinate plane by moving one unit in the positive x or y direction at a time, such that Nick changes direction an odd number of times?

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5 A bag contains nine blue marbles, ten ugly marbles, and one special marble. Ryan picks marbles randomly from this bag with replacement until he draws the special marble. He notices that none of the marbles he drew were ugly. Given this information, what is the expected value of the number of total marbles he drew?

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6 Sarah stands at $(0,0)$ and Rachel stands at $(6,8)$ in the Euclidena plane. Sarah can only move 1 unit in the positive x or y direction, and Rachel can only move 1 unit in the negative x or y direction. Each second, Sarah and Rachel see each other, independently pick a direction to move, and move to their new position. Sarah catches Rachel if Sarah and Rachel are every at the same point. Rachel wins if she is able to get $(0,0)$ without being caught; otherwise, Sarah wins. Given that both of them play optimally to maximize their probability of winning, what is the probability that Rachel wins?

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7 A tourist is learning an incorrect way to sort a permutation $(p_1,\dots,p_n)$ of the integers $(1,\dots,n)$. We define a *fix* on two adjacent elements p_i and p_{i+1}, to be an operation which swaps the two elements if $p_i > p_{i+1}$, and does nothing otherwise. The tourist performs $n-1$ rounds of fixes, numbered $a = 1,2,\dots,n-1$. In round a of fixes, the tourist fixes p_a and p_{a+1}, then p_{a+1} and p_{a+2}, and so on, up to p_{n-1} and p_n. In this process, there are $(n-1)+(n-2)+\cdots+1 = \frac{n(n-1)}{2}$ total fixes performed. How many permutations of $(1,\dots,2018)$ can the tourist start with to obtain $(1,\dots,2018)$ after performing these steps?

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8 A permutation of $\{1,2,\dots,7\}$ is chosen uniformly at random. A partition of the permutation into contiguous blocks is correct if, when each block is sorted independently, the entire permutation becomes sorted. For example, the permutation $(3,4,2,1,6,5,7)$ can be partitioned correctly into the blocks $[3,4,2,1]$ and $[6,5,7]$, since when these blocks are sorted, the permutation becomes $(1,2,3,4,5,6,7)$. Find the expected value of the maximum number of blocks into which the permutation can be partioned correctly.

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9 How many ordered sequences of 36 digits have the property that summing the digits to get a number and taking the last digit of the sum results in a digit which is not in our original sequence? (Digits range from 0 to 9.)

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10 Lily has a 300×300 grid of squares. She now removes 100×100 squares from each of the four corners and colors each of the remaining 50000 squares black and white. Given that no 2×2 square is colored in a checkerboard pattern, find the maximum possible number of (unordered) pairs of squares such that one is black, one is white and the squares share an edge.

– Geometry

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1 Triangle GRT has $GR = 5, RT = 12$, and $GT = 13$. The perpendicular bisector of GT intersects the extension of GR at O. Find TO.

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2 Points A, B, C, D are chosen in the plane such that segments AB, BC, CD, DA have lengths $2, 7, 5, 12$, respectively. Let m be the minimum possible value of the length of segment AC and let M be the maximum possible value of the length of segment AC. What is the ordered pair (m, M)?

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3 How many noncongruent triangles are there with one side of length 20, one side of length 17, and one 60° angle?

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4 A paper equilateral triangle of side length 2 on a table has vertices labeled A, B, C. Let M be the point on the sheet of paper halfway between A and C. Over time, point M is lifted upwards, folding the triangle along segment BM, while A, B, and C on the table. This continues until A and C touch. Find the maximum volume of tetrahedron $ABCM$ at any time during this process.

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5 In the quadrilateral $MARE$ inscribed in a unit circle ω, AM is a diameter of ω, and E lies on the angle bisector of $\angle RAM$. Given that triangles RAM and REM have the same area, find the area of quadrilateral $MARE$.

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6 Let ABC be an equilateral triangle of side length 1. For a real number $0 < x < 0.5$, let A_1 and A_2 be the points on side BC such that $A_1B = A_2C = x$, and let $T_A = \triangle AA_1A_2$. Construct triangles $T_B = \triangle BB_1B_2$ and $T_C = \triangle CC_1C_2$ similarly.

There exist positive rational numbers b, c such that the region of points inside all three triangles T_A, T_B, T_C is a hexagon with area

$$\frac{8x^2 - bx + c}{(2-x)(x+1)} \cdot \frac{\sqrt{3}}{4}.$$

Find (b, c).

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7 Triangle ABC has sidelengths $AB = 14, AC = 13$, and $BC = 15$. Point D is chosen in the interior of $\overline{AB}$ and point E is selected uniformly at random from $\overline{AD}$. Point F is then defined to be the intersection point of the perpendicular to $\overline{AB}$ at E and the union of segments $\overline{AC}$ and $\overline{BC}$. Suppose that D is chosen such that the expected value of the length of $\overline{EF}$ is maximized. Find AD.

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8 Let ABC be an equilateral triangle with side length 8. Let X be on side AB so that $AX = 5$ and Y be on side AC so that $AY = 3$. Let Z be on side BC so that AZ, BY, CX are concurrent. Let ZX, ZY intersect the circumcircle of AXY again at P, Q respectively. Let XQ and YP intersect at K. Compute $KX \cdot KQ$.

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9 Po picks 100 points $P_1, P_2, \cdots, P_{100}$ on a circle independently and uniformly at random. He then draws the line segments connecting $P_1P_2, P_2P_3, \ldots, P_{100}P_1$. Find the expected number of regions that have all sides bounded by straight lines.

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10 Let ABC be a triangle such that $AB = 6, BC = 5, AC = 7$. Let the tangents to the circumcircle of ABC at B and C meet at X. Let Z be a point on the circumcircle of ABC. Let Y be the foot of the perpendicular from X to CZ. Let K be the intersection of the circumcircle of BCY with line AB. Given that Y is on the interior of segment CZ and $YZ = 3CY$, compute AK.

– Team

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1 In an $n \times n$ square array of 1×1 cells, at least one cell is colored pink. Show that you can always divide the square into rectangles along cell borders such that each rectangle contains exactly one pink cell.

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2 Is the number

$$\left(1+\frac{1}{2}\right)\left(1+\frac{1}{4}\right)\left(1+\frac{1}{6}\right)\cdots\left(1+\frac{1}{2018}\right)$$

greater than, less than, or equal to 50?

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3 Michelle has a word with 2^n letters, where a word can consist of letters from any alphabet. Michelle performs a swicheroo on the word as follows: for each $k = 0, 1, \ldots, n-1$, she switches the first 2^k letters of the word with the next 2^k letters of the word. For example, for $n = 3$, Michelle changes

$$ABCDEFGH \to BACDEFGH \to CDBAEFGH \to EFGHCDBA$$

in one switcheroo.

In terms of n, what is the minimum positive integer m such that after Michelle performs the switcheroo operation m times on any word of length 2^n, she will receive her original word?

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4 In acute triangle ABC, let D, E, and F be the feet of the altitudes from A, B, and C respectively, and let L, M, and N be the midpoints of BC, CA and AB, respectively. Lines DE and NL intersect at X, lines DF and LM intersect at Y, and lines XY and BC intersect at Z. Find $\frac{ZB}{ZC}$ in terms of AB, AC, and BC.

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5 Is it possible for the projection of the set of points (x, y, z) with $0 \leq x, y, z \leq 1$ onto some two-dimensional plane to be a simple convex pentagon?

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6 Let $n \geq 2$ be a positive integer. A subset of positive integers S is said to be *comprehensive* if for every integer $0 \leq x < n$, there is a subset of S whose sum has remainder x when divided by n. Note that the empty set has sum 0. Show that if a set S is comprehensive, then there is some (not necessarily proper) subset of S with at most $n-1$ elements which is also comprehensive.

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7 Let $[n]$ denote the set of integers $\{1, 2, \ldots, n\}$. We randomly choose a function $f : [n] \to [n]$, out of the n^n possible functions. We also choose an integer a uniformly at random from $[n]$. Find the probability that there exist positive integers $b, c \geq 1$ such that $f^b(1) = a$ and $f^c(a) = 1$. ($f^k(x)$ denotes the result of applying f to x k times.)

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8 Allen plays a game on a tree with $2n$ vertices, each of whose vertices can be red or blue. Initially, all of the vertices of the tree are colored red. In one move, Allen is allowed to take two vertices of the same color which are connected by an edge and change both of them to the opposite color. He wins if at any time, all of the verices of the tree are colored blue.

(a) Show that Allen can win if and only if the vertices can be split up into two groups V_1 and V_2 to size n, such that each edge in the tree has one endpoint in V_1 and one endpoint in V_2.

(b) Let $V_1 = \{a_1, \ldots, a_n\}$ and $V_2 = \{b_1, \ldots, b_n\}$ from part (a). Let M be the minimum over all permutations σ of $\{1, \ldots, n\}$ of the quantity

$$\sum_{i=1}^{n} d(a_i, b_{\sigma(i)}),$$

where $d(v, w)$ denotes the number of edges along the shortest path between vertices v and w in the tree.
Show that if Allen can win, then the minimum number of moves that it can take for Allen to win is equal to M.

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9 Evan has a simple graph with v vertices and e edges. Show that he can delete at least $\frac{e-v+1}{2}$ edges so that each vertex still has at least half of its original degree.

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10 Let n and m be positive integers in the range $[1, 10^{10}]$. Let R be the rectangle with corners at $(0,0), (n,0), (n,m), (0,m)$ in the coordinate plane. A simple non-self-intersecting quadrilateral with vertices at integer coordinates is called *far-reaching* if each of its vertices lie on or inside R, but each side of R contains at least one vertex of the quadrilateral. Show that there is a far-reaching quadrilateral with area at most 10^6.

– November General

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1 What is the largest factor of 130000 that does not contain the digit 0 or 5?

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2 Twenty-seven players are randomly split into three teams of nine. Given that Zack is on a different team from Mihir and Mihir is on a different team from Andrew, what is the probability that Zack and Andrew are on the same team?

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3 A square in the *xy*-plane has area *A*, and three of its vertices have *x*-coordinates $2, 0,$ and 18 in some order. Find the sum of all possible values of *A*.

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4 Find the number of eight-digit positive integers that are multiples of 9 and have all distinct digits.

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5 Compute the smallest positive integer n for which

$$\sqrt{100+\sqrt{n}}+\sqrt{100-\sqrt{n}}$$

is an integer.

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6 Call a polygon *normal* if it can be inscribed in a unit circle. How many non-congruent normal polygons are there such that the square of each side length is a positive integer?

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7 Anders is solving a math problem, and he encounters the expression $\sqrt{15!}$. He attempts to simplify this radical as $a\sqrt{b}$ where a and b are positive integers. The sum of all possible values of ab can be expressed in the form $q\cdot 15!$ for some rational number q. Find q.

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8 Equilateral triangle ABC has circumcircle Ω. Points D and E are chosen on minor arcs AB and AC of Ω respectively such that $BC=DE$. Given that triangle ABE has area 3 and triangle ACD has area 4, find the area of triangle ABC.

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9 20 players are playing in a Super Mario Smash Bros. Melee tournament. They are ranked $1-20$, and player n will always beat player m if $n<m$. Out of all possible tournaments where each player plays 18 distinct other players exactly once, one is chosen uniformly at random. Find the expected number of pairs of players that win the same number of games.

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10 Real numbers $x, y,$ and z are chosen from the interval $[-1,1]$ independently and uniformly at random. What is the probability that

$$|x|+|y|+|z|+|x+y+z|=|x+y|+|y+z|+|z+x|?$$

– November Theme

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1 Square $CASH$ and regular pentagon $MONEY$ are both inscribed in a circle. Given that they do not share a vertex, how many intersections do these two polygons have?

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2 Consider the addition problem: $\begin{array}{ccccc} & C & A & S & H \\ + & & & M & E \\ \hline O & S & I & D & E \end{array}$ where each letter represents a base-ten digit, and $C, M, O \neq 0.$ (Distinct letters are allowed to represent

the same digit.) How many ways are there to assign values to the letters so that the addition problem
is true?

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3 $HOW, BOW,$ and DAH are equilateral triangles in a plane such that $WO = 7$ and $AH = 2$. Given that D, A, B are collinear in that order, find the length of BA.

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4 I have two cents and Bill has n cents. Bill wants to buy some pencils, which come in two different packages. One package of pencils costs 6 cents for 7 pencils, and the other package of pencils costs a *dime for a dozen* pencils (i.e. 10 cents for 12 pencils). Bill notes that he can spend **all** n of his cents on some combination of pencil packages to get P pencils. However, if I *give my two cents* to Bill, he then notes that he can instead spend **all** $n+2$ of his cents on some combination of pencil packages to get fewer than P pencils. What is the smallest value of n for which this is possible?

Note: Both times Bill must spend **all** of his cents on pencil packages, i.e. have zero cents after either purchase.

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5 Lil Wayne, the rain god, determines the weather. If Lil Wayne makes it rain on any given day, the probability that he makes it rain the next day is 75%. If Lil Wayne doesn't make it rain on one day, the probability that he makes it rain the next day is 25%. He decides not to make it rain today. Find the smallest positive integer n such that the probability that Lil Wayne *makes it rain* n days from today is greater than 49.9%.

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6 Farmer James invents a new currency, such that for every positive integer $n \leq 6$, there exists an n-coin worth $n!$ cents. Furthermore, he has exactly n copies of each n-coin. An integer k is said to be *nice* if Farmer James can make k cents using at least one copy of each type of coin. How many positive integers less than 2018 are nice?

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7 Ben "One Hunna Dolla" Franklin is flying a kite $KITE$ such that IE is the perpendicular bisector of KT. Let IE meet KT at R. The midpoints of KI, IT, TE, EK are A, N, M, D, respectively. Given that $[MAKE] = 18, IT = 10, [RAIN] = 4$, find $[DIME]$.

Note: $[X]$ denotes the area of the figure X.

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8 Crisp All, a basketball player, is *dropping dimes* and nickels on a number line. Crisp drops a dime on every positive multiple of 10, and a nickel on every multiple of 5 that is not a multiple of 10. Crisp then starts at 0. Every second, he has a $\frac{2}{3}$ chance of jumping from his current location x to $x+3$, and a $\frac{1}{3}$ chance of jumping from his current location x to $x+7$. When Crisp jumps on either a dime or a nickel, he stops jumping. What is the probability that Crisp *stops on a dime*?

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9 Circle ω_1 of radius 1 and circle ω_2 of radius 2 are concentric. Godzilla inscribes square $CASH$ in ω_1 and regular pentagon $MONEY$ in ω_2. It then writes down all 20 (not necessarily distinct) distances between a vertex of $CASH$ and a vertex of $MONEY$ and multiplies them all together. What is the maximum possible value of his result?

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10 One million *bucks* (i.e. one million male deer) are in different cells of a 1000×1000 grid. The left and right edges of the grid are then glued together, and the top and bottom edges of the grid are glued together, so that the grid forms a doughnut-shaped torus. Furthermore, some of the bucks are *honest bucks*, who always tell the truth, and the remaining bucks are *dishonest bucks*, who never tell the truth.
Each of the million *bucks* claims that at most one of my neighboring bucks is an *honest buck*. A pair of *neighboring bucks* is said to be *buckaroo* if exactly one of them is an *honest buck* . What is the minimum possible number of *buckaroo* pairs in the grid?

Note: Two *bucks* are considered to be *neighboring* if their cells (x_1, y_1) and (x_2, y_2) satisfy either: $x_1 = x_2$ and $y_1 - y_2 \equiv \pm 1 \pmod{1000}$, or $x_1 - x_2 \equiv \pm 1 \pmod{1000}$ and $y_1 = y_2$.

– November Team

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1 Four standard six-sided dice are rolled. Find the probability that, for each pair of dice, the product of the two numbers rolled on those dice is a multiple of 4.

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2 Alice starts with the number 0. She can apply 100 operations on her number. In each operation, she can either add 1 to her number, or square her number. After applying all operations, her score is the minimum distance from her number to any perfect square. What is the maximum score she can attain?

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3 For how many positive integers $n \leq 100$ is it true that $10n$ has exactly three times as many positive divisors as n has?

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4 Let a and b be real numbers greater than 1 such that $ab = 100$. The maximum possible value of $a^{(\log_{10} b)^2}$ can be written in the form 10^x for some real number x. Find x.

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5 Find the sum of all positive integers n such that $1 + 2 + \cdots + n$ divides

$$15\left[(n+1)^2 + (n+2)^2 + \cdots + (2n)^2\right].$$

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6 Triangle $\triangle PQR$, with $PQ = PR = 5$ and $QR = 6$, is inscribed in circle ω. Compute the radius of the circle with center on $\overline{QR}$ which is tangent to both ω and $\overline{PQ}$.

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7 A 5×5 grid of squares is filled with integers. Call a rectangle *corner-odd* if its sides are grid lines and the sum of the integers in its four corners is an odd number. What is the maximum possible number of corner-odd rectangles within the grid?

Note: A rectangles must have four distinct corners to be considered *corner-odd*; i.e. no $1 \times k$ rectangle can be *corner-odd* for any positive integer k.

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8 Tessa has a unit cube, on which each vertex is labeled by a distinct integer between 1 and 8 inclusive. She also has a deck of 8 cards, 4 of which are black and 4 of which are white. At each step she draws a card from the deck, and-if the card is black, she simultaneously replaces the number on each vertex by the sum of the three numbers on vertices that are distance 1 away from the vertex;-if the card is white, she simultaneously replaces the number on each vertex by the sum of the three numbers on vertices that are distance $\sqrt{2}$ away from the vertex.When Tessa finishes drawing all cards of the deck, what is the maximum possible value of a number that is on the cube?

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9 Let A, B, C be points in that order along a line, such that $AB = 20$ and $BC = 18$. Let ω be a circle of nonzero radius centered at B, and let ℓ_1 and ℓ_2 be tangents to ω through A and C, respectively. Let K be the intersection of ℓ_1 and ℓ_2. Let X lie on segment $\overline{KA}$ and Y lie on segment $\overline{KC}$ such that $XY \| BC$ and XY is tangent to ω. What is the largest possible integer length for XY?

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10 David and Evan are playing a game. Evan thinks of a positive integer N between 1 and 59, inclusive, and David tries to guess it. Each time David makes a guess, Evan will tell him whether the guess is greater than, equal to, or less than N. David wants to devise a strategy that will guarantee that he knows N in five guesses. In David's strategy, each guess will be determined only by Evan's responses to any previous guesses (the first guess will always be the same), and David will only guess a number which satisfies each of Evan's responses. How many such strategies are there?

Note: David need not guess N within his five guesses; he just needs to know what N is after five guesses.
