2015 Pan-African Mathematics Olympiad

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by DylanN

Problem 1 Prove that

$$\sqrt{x-1}+\sqrt{2x+9}+\sqrt{19-3x}<9$$

for all real x for which the left-hand side is well defined.

Problem 2 A convex hexagon $ABCDEF$ is such that

$$AB = BC \quad CD = DE \quad EF = FA$$

and

$$\angle ABC = 2\angle AEC \quad \angle CDE = 2\angle CAE \quad \angle EFA = 2\angle ACE$$

Show that AD, CF and EB are concurrent.

Problem 3 Let $a_1, a_2, ..., a_{11}$ be integers. Prove that there are numbers $b_1, b_2, ..., b_{11}$, each b_i equal $-1, 0$ or 1, but not all being 0, such that the number

$$N = a_1b_1 + a_2b_2 + ... + a_{11}b_{11}$$

is divisible by 2015.

Problem 4 For a positive integer n denote $d(n)$ its greatest odd divisor. Find the value of the sum

$$d(1008) + d(1009) + ... + d(2015)$$

Problem 5 There are seven cards in a hat, and on the card k there is a number 2^{k-1}, $k = 1, 2, ..., 7$. Solarin picks the cards up at random from the hat, one card at a time, until the sum of the numbers on cards in his hand exceeds 124. What is the most probable sum he can get?

Problem 6 Let $ABCD$ be a quadrilateral (with non-perpendicular diagonals).

The perpendicular from A to BC meets CD at K.
The perpendicular from A to CD meets BC at L.
The perpendicular from C to AB meets AD at M.
The perpendicular from C to AD meets AB at N.

1. Prove that KL is parallel to MN.
2. Prove that $KLMN$ is a parallelogram if $ABCD$ is cyclic.
