Thailand Mathematical Olympiad 2015

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– Day 1

1 Let p be a prime, and let $a_1, a_2, a_3, ...$ be a sequence of positive integers so that $a_n a_{n+2} = a_{n+1}^2 + p$ for all positive integers n. Show that a_{n+1} divides $a_n + a_{n+2}$ for all positive integers n.

2 Let a, b, c be positive reals with $abc = 1$. Prove the inequality

$$\frac{a^5}{a^3+1} + \frac{b^5}{b^3+1} + \frac{c^5}{c^3+1} \geq \frac{3}{2}$$

and determine all values of a, b, c for which equality is attained

3 Let $P = \{(x, y)|x, y \in \{0, 1, 2, ..., 2015\}\}$ be a set of points on the plane. Straight wires of unit length are placed to connect points in P so that each piece of wire connects exactly two points in P, and each point in P is an endpoint of exactly one wire. Prove that no matter how the wires are placed, it is always possible to draw a straight line parallel to either the horizontal or vertical axis passing through midpoints of at least 506 pieces of wire.

4 Let $\triangle\ ABC$ be a triangle with an obtuse angle $\angle ACB$. The incircle of $\triangle\ ABC$ centered at I is tangent to the sides AB, BC, CA at D, E, F respectively. Lines AI and BI intersect EF at M and N respectively. Let G be the midpoint of AB. Show that M, N, G, D lie on a circle.

5 Let n be an integer greater than 6.Show that if $n+1$ is a prime number,than $\left\lceil \frac{(n-1)!}{n(n+1)} \right\rceil$ is ODD.

– Day 2

6 Let m and n be positive integers. Determine the number of ways to fill each cell of an $m \times n$ table with a number from $\{-2, -1, 1, 2\}$ so that the product of the numbers written in each row and column is -2.

7 Let A, B, C be centers of three circles that are mutually tangent externally, let r_A, r_B, r_C be the radii of the circles, respectively. Let r be the radius of the incircle of $\triangle\ ABC$. Prove that

$$r^2 \leq \frac{1}{9}(r_A^2 + r_B^2 + r_C^2)$$

and identify, with justification, one case where the equality is attained.

8 Let m and n be positive integers such that $m-n$ is odd. Show that $(m+3n)(5m+7n)$ is not a perfect square.

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9 Determine all functions $f:R\to R$ satisfying $f(f(x)+2y)=6x+f(f(y)-x)$ for all real numbers x,y

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10 A Boy Scouts camp holds a campfire. The camp has scarfs of m colors with n scarves of each color, and gives each of its mn scouts a scarf, where $m,n\ge 2$ are integers. The camp then divides its scouts into troops by the color of their scarfs.
At the beginning of the campfire, the scouts are seated in a circle so that scouts in the same troop are seated next to each other.
The camp organizer then proceeds to select, round by round, representatives to perform a show, with the following conditions: there must be at least two representatives in each round, they must come from the same troop, and any specific set of representatives can only perform once. (For example, if $\{A,B\}$ has performed, then $\{A,B\}$ cannot perform again, but $\{A,B,C\}$ can still perform.) This process is repeated until all valid sets of representatives have performed.
At this point, the organizers order each scout to hand their scarfs to the scout to the left, and re-group the scouts into troops, again according to their scarf color, and the process above is resumed, until the set of valid sets of representatives is exhausted again. (The sets of representatives after re-grouping must also be distinct from the sets before re-grouping.)
When that happens, the organizers order another re-group, and resumes the process, and this repeats until there can be no further performances. Find, in simple form, the total number of performances that will be performed.
