**Thailand Mathematical Olympiad 2013**

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| – | Day 1 |
| 1 | Find the largest integer that divides $p^4 - 1$ for all primes $p > 4$ |
| 2 | Let $\triangle\ ABC$ be a triangle with $\angle ABC > \angle BCA \geq 30^o$ . The angle bisectors of $\angle ABC$ and $\angle BCA$ intersect $CA$ and $AB$ at $D$ and $E$ respectively, and $BD$ and $CE$ intersect at $P$. Suppose that $PD = PE$ and the incircle of $\triangle\ ABC$ has unit radius. What is the maximum possible length of $BC$? |
| 3 | Each point on the plane is colored either red or blue. Show that there are three points of the same color that form a triangle with side lengths $1, 2, \sqrt{3}$. |
| 4 | Determine all monic polynomials $p(x)$ having real coefficients and satisfying the following two conditions: • $p(x)$ is nonconstant, and all of its roots are distinct reals • If $a$and $b$ are roots of $p(x)$ then $a + b + ab$ is also a root of $p(x)$. |
| 5 | Find a five-digit positive integer $n$ (in base $10$) such that $n^3 - 1$ is divisible by $2556$ and which minimizes the sum of digits of $n$. |
| 6 | Determine all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying $(x^2 + y^2)f(xy) = f(x)f(y)f(x^2 + y^2)\ \forall x, y \in \mathbb{R}$ |
| – | Day 2 |
| 7 | Let $P_1, ..., P_{2556}$ be distinct points in a regular hexagon $ABCDEF$ with unit side length. Suppose that no three points in the set $S = \{A, B, C, D, E, F, P_1, ..., P_{2556}\}$ are collinear. Show that there is a triangle whose vertices are in $S$ and whose area is less than $\frac{1}{1700}$ . |
| 8 | Let $p(x) = x^{2013} + a_{2012}x^{2012} + a_{2011}x^{2011} + ... + a_1x + a_0$ be a polynomial with real coefficients with roots $-b_{1006}, -b_{1005}, ..., -b_1, 0, b_1, ..., b_{1005}, b_{1006}$, where $b_1, b_2, ..., b_{1006}$ are positive reals with product $1$. Show that $a_3a_{2011} \leq 1012036$ |
| 9 | Let $ABCD$ be a convex quadrilateral, and let $M$ and$N$ be midpoints of sides $AB$ and $CD$ respectively. Point $P$ is chosen on $CD$ so that $MP \perp CD$, and point $Q$ is chosen on $AB$ so that $NQ \perp AB$. Show that $AD \parallel BC$ if and only if $\frac{AB}{CD} = \frac{MP}{NQ}$ . |
| 10 | Find all pairs of positive integers $(x, y)$ such that $\frac{xy^3}{x+y}$ is the cube of a prime. |

**11** Let $m, n$ be positive integers.
There are $n$ piles of gold coins, so that pile $i$ has $a_i > 0$ coins in it $(i = 1, ..., n)$. Consider the following game:

Step 1. Nadech picks sets $B_1, B_2, ..., B_n$, where each $B_i$ is a nonempty subset of $\{1, 2, ..., m\}$, and gives them to Yaya.

Step 2. Yaya picks a set $S$ which is also a nonempty subset of $\{1, 2, ..., m\}$.

Step 3. For each $i = 1, 2, ..., n$, Nadech wins the coins in pile $i$ if $B_i \cap S$ has an even number of elements, and Yaya wins the coins in pile $i$ if $B_i \cap S$ has an odd number of elements.

Show that, no matter how Nadech picks the sets $B_1, B_2, ..., B_n$, Yaya can always pick $S$ so that she ends up with more gold coins than Nadech

**12** Let $\omega$ be the incircle of $\triangle ABC$, $\omega$ is tangent to sides $BC$ and $AC$ at $D$ and $E$ respectively. The line perpendicular to $BC$ at $D$ intersects $\omega$ again at $P$. Lines $AP$ and $BC$ intersect at $M$. Let $N$ be a point on segment $AC$ so that $AE = CN$. Line $BN$ intersects $\omega$ at $Q$ (closer to $B$) and intersect $AM$ at $R$. Show that the area of $\triangle ABR$ is equal to the area of $PQMN$.