Thailand Mathematical Olympiad 2012

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–	Day 1
1	Let $\triangle ABC$ be a right triangle with $\angle B = 90^o$. Let P be a point on side BC, and let ω be the circle with diameter CP. Suppose that ω intersects $AC$and AP again at Q and R, respectively. Show that $CP^2 = AC \cdot CQ - AP \cdot PR$.
2	Let $a_1, a_2, ..., a_{2012}$ be pairwise distinct integers. Show that the equation $(x - a_1)(x - a_2)...(x - a_{2012}) = (1006!)^2$ has at most one integral solution.
3	Let $m, n > 1$ be coprime odd integers. Show that $$\lfloor \frac{m^{\phi(n)+1} + n^{\phi(m)+1}}{mn} \rfloor$$ is an even integer, where ϕ is Eulers totient function.
4	Let $ABCD$ be a unit square. Points E, F, G, H are chosen outside $ABCD$ so that $\angle AEB = \angle BFC = \angle CGD = \angle DHA = 90^o$. Let O_1, O_2, O_3, O_4, respectively, be the incenters of $\triangle ABE, \triangle BCF, \triangle CDG, \triangle DAH$. Show that the area of $O_1O_2O_3O_4$ is at most 1.
5	Determine all functions $f : R \to R$ satisfying $f(f(x)+xf(y)) = 3f(x)+4xy$ for all real numbers x, y.
–	Day 2
7	Let a, b, m be integers such that gcd $(a, b) = 1$ and $5 \mid ma^2+b^2$. Show that there exists an integer n such that $5 \mid m - n^2$.
8	$4n$ first grade students at Songkhla Primary School, including $2n$ boys and $2n$ girls, participate in a taekwondo tournament where every pair of students compete against each other exactly once. The tournament is scored as follows: • In a match between two boys or between two girls, a win is worth 3 points, a draw 1 point, and a loss 0 points. • In a math between a boy and a girl, if the boy wins, he receives 2 points, else he receives 0 points. If the girl wins, she receives 3 points, if she draws, she receives 2 points, and if she loses, she receives 0 points. After the tournament, the total score of each student is calculated. Let P be the number of matches ending in a draw, and let Q be the total number of matches. Suppose that the maximum total score is $4n - 1$. Find P/Q.

9 Let n be a positive integer and let $P(x) = x^n + a_{n-1}x^{n-1} + ... + a_1x + 1$ be a polynomial with positive real coefficients. Under the assumption that the roots of P are all real, show that $P(x) \geq (x+1)^n$ for all $x > 0$.

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10 Let x be an irrational number. Show that there are integers m and n such that $\frac{1}{2555} < mx + n < \frac{1}{2012}$

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11 Let $\triangle ABC$ be an acute triangle, and let P be the foot of altitude from C to AB. Let ω be the circle with diameter BC. The tangents from A to ω are drawn touching ω at D and E. Lines AD and AE intersect line BC at M and N respectively, so that B lies between M and C. Let CP intersect DE at Q, ME intersect ND at R, and let QR intersect BC at S. Show that QS bisects $\angle DSE$

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12 Let a, b, c be positive integers. Show that if $\frac{a}{b} + \frac{b}{c} + \frac{c}{a}$ is an integer then abc is a perfect cube.
