Thailand Mathematical Olympiad 2004 - then it was named as 1st POSN Mathematical Olympiad

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– Day 1

1 Given that $\cos 4A = \frac{1}{3}$ and $-\frac{\pi}{4} \le A \le \frac{\pi}{4}$, find the value of $\cos^8 A - \sin^8 A$.

2 Let a and b be real numbers such that

$$\begin{cases} a^6 - 3a^2b^4 = 3 \\ b^6 - 3a^4b^2 = 3\sqrt{2}. \end{cases}$$

What is the value of $a^4 + b^4$?

3 Let u, v, w be the roots of $x^3 - 5x^2 + 4x - 3 = 0$. Find a cubic polynomial having u^3, v^3, w^3 as roots.

4 Find all real solutions x to the equation

$$x = \sqrt{x - \frac{1}{x}} + \sqrt{1 - \frac{1}{x}}$$

5 Let n be a given positive integer. Find the solution set of the equation $\sum_{k=1}^{2n} \sqrt{x^2 - 2kx + k^2} = |2nx - n - 2n^2|$

6 Let $f(x) = x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$. Find the remainder when $f(x^7)$ is divided by $f(x)$.

7 Let f be a function such that $f(0) = 0, f(1) = 1$, and $f(n) = 2f(n-1) - f(n-2) + (-1)^n(2n-4)$ for all integers $n \ge 2$. Find f(n) in terms of n.

8 Let $f : R \to R$ satisfy $f(x + f(y)) = 2x + 4y + 2547$ for all reals x, y. Compute $f(0)$.

9 Compute the sum

$$\sum_{k=0}^{n} \frac{(2n)!}{k!^2(n-k)!^2}.$$

10 Find the number of ways to select three distinct numbers from $1, 2, ..., 3n$ with a sum divisible by 3.

11 Find the number of positive integer solutions to $(x_1 + x_2 + x_3)(y_1 + y_2 + y_3 + y_4) = 91$

12 Let n be a positive integer and define $A_n = \{1, 2, ..., n\}$. How many functions $f : A_n \to A_n$ are there such that for all $x, y \in A_n$, if $x < y$ then $f(x) \geq f(y)$?

13 Compute the remainder when $29^{30} + 31^{28} + 28! \cdot 30!$ is divided by $29 \cdot 31$.

14 Compute gcd$(5^{2547} - 1, 5^{2004} - 1)$.

15 Find the largest positive integer $n \leq 2004$ such that $3^{3n+3} - 27$ is divisible by 169.

16 What are last three digits of $2^{2^{2004}}$?

17 Compute the remainder when $1^{2547} + 2^{2547} + ... + 2547^{2547}$ is divided by 25.

18 Find positive reals a, b, c which maximizes the value of abc subject to the constraint that $b(a^2 + 2) + c(a + 2) = 12$.

19 Find positive reals a, b, c which maximizes the value of $a + 2b + 3c$ subject to the constraint that $9a^2 + 4b^2 + c^2 = 91$

20 Two pillars of height a and b are erected perpendicular to the ground. On each pillar, a straight cable is placed connecting the top of the pillar to the base of the other pillar; the two lines of cable intersect at a point above ground. What is the height of this point?

21 The ratio between the circumradius and the inradius of a given triangle is $7 : 2$. If the length of two sides of the triangle are 3 and 7, and the length of the remaining side is also an integer, what is the length of the remaining side?

– Day 2 (proof based)

1 A $\triangle\ ABC$ is given with $\angle A = 70^o$. The angle bisectors of $\triangle\ ABC$ intersect at I. Suppose that $CA + AI = BC$. Find, with proof, the value of $\angle B$.

2 Let $f : Q \to Q$ be a function satisfying the equation $f(x+y) = f(x)+f(y)+2547$ for all rational numbers x, y. If $f(2004) = 2547$, find $f(2547)$.

3 18 students with pairwise distinct heights line up. Ideally, the teacher wants the students to be ordered by height so that the tallest student is in the back of the line. However, it turns out that this is not the case, so when the teacher sees two consecutive students where the taller of the two is in front, the two students are swapped. It turns out that 150 swaps must be made before the students are lined up in the correct order. How many possible starting orders are there?

4 Let $ABCD$ be a convex quadrilateral. Prove that area $(ABCD) \leq \frac{AB^2+BC^2+CD^2+DA^2}{4}$

5 Find all primes p such that $p^2 + 2543$ has at most 16 divisors.

6 Let $a, b, c > 0$ satisfy $a + b + c \geq \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$. Prove that $a^3 + b^3 + c^3 \geq a + b + c$