Thailand Mathematical Olympiad 2005 - then it was named as 2nd POSN Mathematical Olympiad

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–	Day 1
1	Let $ABCD$ be a trapezoid inscribed in a unit circle with diameter AB. If $DC = 4AD$, compute AD.
2	Let $\triangle ABC$ be an acute triangle, and let A' and B' be the feet of altitudes from A to BC and from B to CA, respectively; the altitudes intersect at H. If BH is equal to the circumradius of $\triangle ABC$, find $\frac{A'B}{AB}$.
3	Triangle $\triangle ABC$ is isosceles with $AB = AC$ and $\angle ABC = 2\angle BAC$. Compute $\frac{AB}{BC}$.
4	Triangle $\triangle ABC$ is inscribed in the circle with diameter BC. If $AB = 3$, $AC = 4$, and O is the incenter of $\triangle ABC$, then find $BO \cdot OC$.
5	A die is thrown six times. How many ways are there for the six rolls to sum to 21?
6	Find the number of positive integer solutions to the equation $(x_1 + x_2 + x_3)^2(y_1 + y_2) = 2548$.
7	How many ways are there to express 2548 as a sum of at least two positive integers, where two sums that differ in order are considered different?
8	For each subset T of $S = \{1, 2, ..., 7\}$, the result $r(T)$ of T is computed as follows: the elements of T are written, largest to smallest, and alternating signs $(+, -)$ starting with $+$ are put in front of each number. The value of the resulting expression is$r(T)$. (For example, for $T = \{2, 4, 7\}$, we have $r(T) = +7 - 4 + 2 = 5$.) Compute the sum of $r(T)$ as T ranges over all subsets of S.
9	Compute gcd $\left(\frac{135^{90}-45^{90}}{90^2}, 90^2\right)$
10	What is the remainder when $\sum_{k=1}^{2005} k^{2005\cdot 2^{2005}}$ is divided by 2^{2005}?
11	Find the smallest positive integer x such that 2^{254} divides $x^{2005} + 1$.
12	Find the number of even integers n such that $0 \le n \le 100$ and $5\|n^2 \cdot 2^{2n^2} + 1$.
13	Find all odd integers k for which there exists a positive integer m satisfying the equation $k + (k + 5) + (k + 10) + ... + (k + 5(m - 1)) = 1372$.

14 A function $f : Z \to Z$ is given so that $f(m+n) = f(m) + f(n) + 2mn - 2548$ for all positive integers m, n. Given that $f(2548) = -2548$, find the value of $f(2)$.

15 A function $f : R \to R$ satisfy the functional equation $f(x+2y) + 2f(y-2x) = 3x - 4y + 6$ for all reals x, y. Compute $f(2548)$.

16 Compute the sum of roots of $(2-x)^{2005} + x^{2005} = 0$.

17 For $a, b \geq 0$ we define $a * b = \frac{a+b+1}{ab+12}$. Compute $0 * (1 * (2 * (...(2003 * (2004 * 2005))...)))$.

18 Compute the sum

$$\sum_{k=0}^{1273} \frac{1}{1 + tan^{2548}\left(\frac{k\pi}{2548}\right)}$$

19 Let $P(x)$ be a monic polynomial of degree 4 such that for $k = 1, 2, 3$, the remainder when $P(x)$ is divided by $x - k$ is equal to k. Find the value of $P(4) + P(0)$.

20 Let $a, b, c, d > 0$ satisfy $36a + 4b + 4c + 3d = 25$. What is the maximum possible value of $ab^{1/2}c^{1/3}d^{1/4}$?

21 Compute the minimum value of $cos(a-b) + cos(b-c) + cos(c-a)$ as a, b, c ranges over the real numbers.

– Day 2 (proof based)

1 A point A is chosen outside a circle with diameter BC so that $\triangle\ ABC$ is acute. Segments AB and AC intersect the circle at D and E, respectively, and CD intersects BE at F. Line AF intersects the circle again at G and intersects BC at H. Prove that $AH \cdot FH = GH^2$.
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2 Let Sbe a set of three distinct integers. Show that there are $a, b \in S$ such that $a \neq b$ and $10|a^3b - ab^3$.

3 Does there exist a function $f : Z^+ \to Z^+$ such that $f(f(n)) = 2n$ for all positive integers n? Justify your answer, and if the answer is yes, give an explicit construction.

4 Let O_1 be the center of a semicircle ω_1 with diameter AB and let O_2 be the center of a circle ω_2 inscribed in ω_1 and which is tangent to AB at O_1. Let O_3 be a point on AB that is the center of a semicircle ω_3 which is tangent to both ω_1 and ω_2. Let P be the intersection of the line through O_3 perpendicular to AB and the line through O_2 parallel to AB. Show that P is the center of a circle Γ tangent to all of ω_1, ω_2 and ω_3.

6 Let a, b, c be distinct real numbers. Prove that

$$\left(\frac{2a-b}{a-b}\right)^2+\left(\frac{2b-c}{b-c}\right)^2+\left(\frac{2c-a}{c-a}\right)^2\geq 5$$