Thailand Mathematical Olympiad 2007 - then it was named as 4th POSN Mathematical Olympiad

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– Day 1

1 In a circle $\odot O$, radius OA is perpendicular to radius OB. Chord AC intersects OB at E so that the length of arc AC is one-third the circumference of $\odot O$. Point D is chosen on OB so that $CD \perp AB$. Suppose that segment AC is 2 units longer than segment OD. What is the length of segment AC?

2 Let $ABCD$ be a cyclic quadrilateral so that arcs AB and BC are equal. Given that $AD = 6, BD = 4$ and $CD = 1$, compute AB.

3 A triangle $\triangle\ ABC$ has $\angle B = 90^o$. A circle is tangent to AB at B and also tangent to AC. Another circle is tangent to the first circle as well as the two sides AB and AC. Suppose that $AB = \sqrt{3}$ and $BC = 3$. What is the radius of the second circle?

4 A triangle $\triangle\ ABC$ has $AC = 16$ and $BC = 12$. E and F are points on AC and BC, respectively, so that $CE = 3CF$. Let M be the midpoint of AB, and let lines EF and CM intersect at G. Compute the ratio $EG : GF$.

5 A triangle $\triangle\ ABC$ has $\angle A = 90^o$, and a point D is chosen on AC. Point F is the foot of altitude from A to BC. Suppose that $BD = DC = CF = 2$. Compute AC.

6 Let M be the midpoint of a given segment BC. Point A is chosen to maximize $\angle ABC$ while subject to the condition that $\angle MAC = 20^o$. What is the ratio BC/BA ?

7 Let a, b, c be complex numbers such that $a + b + c = 1$, $a^2 + b^2 + c^2 = 2$ and $a^3 + b^3 + c^3 = 3$. Find the value of $a^4 + b^4 + c^4$.

8 Let $x_1, x_2, ..., x_{84}$ be the roots of the equation $x^{84} + 7x - 6 = 0$. Compute $\sum_{k=1}^{84} \frac{x_k}{x_k-1}$.

9 Let $f : R \to R$ be a function satisfying the equation $f(x^2+x+3)+2f(x^2-3x+5) = 6x^2-10x+17$ for all real numbers x. What is the value of $f(85)$?

10 Find the smallest positive integer n such that the equation $\sqrt{3}z^{n+1} - z^n - 1 = 0$ has a root on the unit circle.

11 Compute the number of functions $f : \{1, 2, ..., 2550\} \to \{61, 80, 84\}$ such that $\sum_{k=1}^{2550} f(k)$ is divisible by 3.

12 An alien with four feet wants to wear four identical socks and four identical shoes, where on each foot a sock must be put on before a shoe. How many ways are there for the alien to wear socks and shoes?

13 Let $S = \{1, 2, ..., 8\}$. How many ways are there to select two disjoint subsets of S?

14 The sum

$$\sum_{k=84}^{8000} \binom{k}{84}\binom{8084-k}{84}$$

can be written as a binomial coefficient $\binom{a}{b}$ for integers a, b. Find a possible pair (a, b)

15 Compute the remainder when $222!^{111} + 111^{222!} + 111!^{222} + 222^{111!}$ is divided by 2007.

16 What is the smallest positive integer with 24 positive divisors?

17 Compute the product of positive integers n such that $n^2 + 59n + 881$ is a perfect square.

18 Let p_k be the k-th prime number. Find the remainder when $\sum_{k=2}^{2550} p_k^{p_k^4-1}$ is divided by 2550.

– Day 2 (proof based)

1 Find all functions $f : R \to R$ such that the inequality

$$\sum_{i=1}^{2549} f(x_i + x_{i+1}) + f(\sum_{i=1}^{2550} x_y) \le \sum_{i=1}^{2550} f(2x_i)$$

for all reals $x_1, x_2, ..., x_{2550}$.

2 In a dance party there are n girls and n boys, and some m songs are played. Each song is danced to by at least one pair of a boy and a girl, who both receive a *malai* each. Prove that for all positive integers $k \le n$, it is possible to select k boys and $n - k$ girls so that the n selected people received at least m malai in total.

3 Two circles intersect at X and Y . The line through the centers of the circles intersect the first circle at A and C, and intersect the second circle at B and D so that A, B, C, D lie in this order. The common chord XY cuts BC at P, and a point O is arbitrarily chosen on segment XP. Lines CO and BO are extended to intersect the first and second circles at M and N, respectively. If lines AM and DN intersect at Z, prove that X, Y and Z lie on the same line.

4 Find all primes p such that $\frac{2^{p-1}-1}{p}$ is a perfect square.

5 The freshman class of a school consists of 229 boys and 271 girls, and is divided into 10 rooms of 50 students each, the students in each room are numbered from 1 to 50. The physical education teacher wants to select a relay running team consisting of 1 boy and 3 girls or 1 girl and 3 boys, so that the four students must be two pairs of students with the same number from two rooms. Show that the number of possible teams is odd.

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6 A triangle has perimeter $2s$, inradius r, and incenter I. If s_a, s_b and s_c are the distances from I to the three vertices, then show that

$$\frac{3}{4}+\frac{r}{s_a}+\frac{r}{s_b}+\frac{r}{s_c} \leq \frac{s^2}{12r^2}$$
