Thailand Mathematical Olympiad 2008 - then it was named as 5st POSN Mathematical Olympiad

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by parmenides51

–	Day 1
1	Let $\triangle ABC$ be a triangle with $\angle BAC = 90^o$ and $\angle ABC = 60^o$. Point E is chosen on side BC so that $BE : EC = 3 : 2$. Compute $\cos \angle CAE$.
2	Let AD be the common chord of two equal-sized circles O_1 and O_2. Let B and C be points on O_1 and O_2, respectively, so that D lies on the segment BC. Assume that $AB = 15, AD = 13$ and $BC = 18$, what is the ratio between the inradii of $\triangle ABD$ and $\triangle ACD$?
3	Find all positive real solutions to the equation $x + \left\lfloor \frac{x}{3} \right\rfloor = \left\lfloor \frac{2x}{3} \right\rfloor + \left\lfloor \frac{3x}{5} \right\rfloor$
4	Prove that $$\sqrt{a^2 + b^2 - \sqrt{2}ab} + \sqrt{b^2 + c^2 - \sqrt{2}bc} \ge \sqrt{a^2 + c^2}$$ for all real numbers $a, b, c > 0$
5	Let $P(x)$ be a polynomial of degree 2008 with the following property: all roots of P are real, and for all real a, if $P(a) = 0$ then $P(a + 1) = 1$. Prove that P must have a repeated root.
6	Let $f : R \to R$ be a function satisfying the inequality $\|f(x + y) - f(x) - f(y)\| < 1$ for all reals x, y. Show that $\left\|f\left(\frac{x}{2008}\right) - \frac{f(x)}{2008}\right\| < 1$ for all real numbers x.
7	Two positive integers m, n satisfy the two equations $m^2+n^2 = 3789$ and $gcd(m, n)+lcm(m, n) = 633$. Compute $m + n$.
8	Prove that $2551 \cdot 543^n - 2008 \cdot 7^n$ is never a perfect square, where n varies over the set of positive integers
9	Find the number of pairs of sets (A, B) satisfying $A \subseteq B \subseteq \{1, 2, ..., 10\}$
10	On the sides of triangle $\triangle ABC$, 17 points are added, so that there are 20 points in total (including the vertices of $\triangle ABC$.) What is the maximum possible number of (nondegenerate) triangles that can be formed by these points.
–	Day 2 (proof based)

1 Let P be a point outside a circle ω. The tangents from P to ω are drawn touching ω at points A and B. Let M and N be the midpoints of AP and AB, respectively. Line MN is extended to cut ω at C so that N lies between M and C. Line PC intersects ω again at D, and lines ND and PB intersect at O. Prove that $MNOP$ is a rhombus.

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2 Find all positive integers N with the following properties:
(i) N has at least two distinct prime factors, and
(ii) if $d_1 < d_2 < d_3 < d_4$ are the four smallest divisors of N then $N = d_1^2 + d_2^2 + d_3^2 + d_4^2$

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3 For each positive integer n, define $a_n = n(n+1)$. Prove that

$$n^{1/a_1} + n^{1/a_3} + n^{1/a_5} + ... + n^{1/a_{2n-1}} \geq n^{a_{3n+2}/a_{3n+1}}$$

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4 Let n be a positive integer. Show that

$$\binom{2n+1}{1} - \binom{2n+1}{3}2008 + \binom{2n+1}{5}2008^2 - ... + (-1)^{2n+1}\binom{2n+1}{2n+1}2008^n$$

is not divisible by 19.

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5 Students in a class consisting of m boys and n girls line up. Over all possible ways of lining up, compute the average number of pairs of two boys or two girls who are next to each other.

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6 Let $f : R^+ \to R^+$ satisfy $f(xy)^2 = f(x^2)f(y^2)$ for all positive reals x, y with $x^2y^3 > 2008$. Prove that $f(xy)^2 = f(x^2)f(y^2)$ for all positive reals x, y.
