

**Thailand Mathematical Olympiad 2010**

[www.artofproblemsolving.com/community/c1259251](http://www.artofproblemsolving.com/community/c1259251)

by parmenides51

– Day 1

- 
- 1** Find the number of ways to distribute 11 balls into 5 boxes with different sizes, so that each box receives at least one ball, and the total number of balls in the largest and smallest boxes is more than the total number of balls in the remaining boxes.
- 
- 2** Let  $\triangle ABC$  be an isosceles triangle with  $AB = AC$ . A circle passing through  $B$  and  $C$  intersects sides  $AB$  and  $AC$  at  $D$  and  $E$  respectively. A point  $F$  on this circle is chosen so that  $EF \perp BC$ . If  $BC = x$ ,  $CF = y$ , and  $BF = z$ , find the length of  $DF$  in terms of  $x, y, z$ .
- 
- 3** Show that there are infinitely many positive integers  $n$  such that  $2 \underbrace{555\dots55}_n 3$  is divisible by 2553.
- 
- 4** Let  $\triangle ABC$  be an equilateral triangle, and let  $M$  and  $N$  be points on  $AB$  and  $AC$ , respectively, so that  $AN = BM$  and  $3MB = AB$ . Lines  $CM$  and  $BN$  intersect at  $O$ . Find  $\angle AOB$ .
- 
- 5** In a round-robin table tennis tournament between 2010 athletes, where each match ends with a winner and a loser, let  $a_1, \dots, a_{2010}$  denote the number of wins of each athlete, and let  $b_1, \dots, b_{2010}$  denote the number of losses of each athlete. Show that  $a_1^2 + a_2^2 + \dots + a_{2010}^2 = b_1^2 + b_2^2 + \dots + b_{2010}^2$ .
- 
- 6** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a function satisfying the functional equation  $f(3x + y) + f(3x - y) = f(x + y) + f(x - y) + 16f(x)$  for all reals  $x, y$ . Show that  $f$  is even, that is,  $f(-x) = f(x)$  for all reals  $x$ .
- 
- 7** Let  $a, b, c$  be positive reals. Show that  $\frac{a^5}{bc^2} + \frac{b^5}{ca^2} + \frac{c^5}{ab^2} \geq a^2 + b^2 + c^2$ .
- 
- 8** Define the modulo 2553 distance  $d(x, y)$  between two integers  $x, y$  to be the smallest nonnegative integer  $d$  equivalent to either  $x - y$  or  $y - x$  modulo 2553. Show that, given a set  $S$  of integers such that  $|S| \geq 70$ , there must be  $m, n \in S$  with  $d(m, n) \leq 36$ .
- 
- 9** Let  $a, b, c$  be real numbers so that all roots of the equation  $2x^5 + 5x^4 + 5x^3 + ax^2 + bx + c = 0$  are real. Find the smallest real root of the equation above.
- 
- 10** Find all primes  $p$  such that  $\binom{100}{p} + 7$  is divisible by  $p$ .

– Day 2

1 Show that, for every positive integer  $x$ , there is a positive integer  $y \in \{2, 5, 13\}$  such that  $xy - 1$  is not a perfect square.

---

2 The Ministry of Education selects 2010 students from 5 regions of Thailand to participate in a debate tournament, where each pair of students will debate in one of the three topics: politics, economics, and societal problems. Show that there are 3 students who were born in the same month, come from the same region, are of the same gender, and whose pairwise debates are on the same topic.

---

3 Let  $\triangle ABC$  be a scalene triangle with  $AB < BC < CA$ . Let  $D$  be the projection of  $A$  onto the angle bisector of  $\angle ABC$ , and let  $E$  be the projection of  $A$  onto the angle bisector of  $\angle ACB$ . The line  $DE$  cuts sides  $AB$  and  $AC$  at  $M$  and  $N$ , respectively. Prove that

$$\frac{AB + AC}{BC} = \frac{DE}{MN} + 1$$

---

4 For  $i = 1, 2$  let  $\triangle A_i B_i C_i$  be a triangle with side lengths  $a_i, b_i, c_i$  and altitude lengths  $p_i, q_i, r_i$ . Define  $a_3 = \sqrt{a_1^2 + a_2^2}$ ,  $b_3 = \sqrt{b_1^2 + b_2^2}$ , and  $c_3 = \sqrt{c_1^2 + c_2^2}$ . Prove that  $a_3, b_3, c_3$  are side lengths of a triangle, and if  $p_3, q_3, r_3$  are the lengths of altitudes of this triangle, then  $p_3^2 \geq p_1^2 + p_2^2$ ,  $q_3^2 \geq q_1^2 + q_2^2$ , and  $r_3^2 \geq r_1^2 + r_2^2$ .

---

5 Determine all functions  $f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  satisfying the equation  $f(x - t, y) + f(x + t, y) + f(x, y - t) + f(x, y + t) = 2010$  for all real numbers  $x, y$  and for all nonzero  $t$ .

---

6 Show that no triples of primes  $p, q, r$  satisfy  $p > r, q > r$ , and  $pq \mid r^p + r^q$ .

---