Silk Road Mathematics Competiton 2020

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1 Given a strictly increasing infinite sequence of natural numbers $a_1, a_2, a_3, \ldots$. It is known that $a_n \leq n + 2020$ and the number $n^3 a_n - 1$ is divisible by a_{n+1} for all natural numbers n. Prove that $a_n = n$ for all natural numbers n.

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2 The triangle ABC is inscribed in the circle ω. Points K, L, M are marked on the sides AB, BC, CA, respectively, and $CM \cdot CL = AM \cdot BL$. Ray LK intersects line AC at point P. The common chord of the circle ω and the circumscribed circle of the triangle KMP meets the segment AM at the point S. Prove that $SK \parallel BC$.

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3 A polynomial $Q(x) = k_n x^n + k_{n-1} x^{n-1} + \ldots + k_1 x + k_0$ with real coefficients is called *powerful* if the equality $|k_0| = |k_1| + |k_2| + \ldots + |k_{n-1}| + |k_n|$, and *non-increasing* , if $k_0 \geq k_1 \geq \ldots \geq k_{n-1} \geq k_n$. Let for the polynomial $P(x) = a_d x^d + a_{d-1} x^{d-1} + \ldots + a_1 x + a_0$ with nonzero real coefficients, where $a_d > 0$, the polynomial $P(x)(x-1)^t (x+1)^s$ is *powerful* for some non-negative integers s and t ($s + t > 0$). Prove that at least one of the polynomials $P(x)$ and $(-1)^d P(-x)$ is *nonincreasing*.

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4 Prove that for any natural number m there exists a natural number n such that any n distinct points on the plane can be partitioned into m non-empty sets whose *convex hulls* have a common point.
The *convex hull* of a finite set X of points on the plane is the set of points lying inside or on the boundary of at least one convex polygon with vertices in X, including degenerate ones, that is, the segment and the point are considered convex polygons. No three vertices of a convex polygon are collinear. The polygon contains its border.
