India International Mathematical Olympiad Training Camp 2015

www.artofproblemsolving.com/community/c126383

by Sayan, hajimbrak

–	Practice Test 1
–	April 26th
1	Find all positive integers a, b such that $\frac{a^2+b}{b^2-a}$ and $\frac{b^2+a}{a^2-b}$ are also integers.
2	A 10-digit number is called a *cute* number if its digits belong to the set $\{1, 2, 3\}$ and the difference of every pair of consecutive digits is 1. a) Find the total number of cute numbers. b) Prove that the sum of all cute numbers is divisibel by 1408.
3	Prove that for any triangle ABC, the inequality $\sum_{\text{cyclic}} \cos A \leq \sum_{\text{cyclic}} \sin(A/2)$ holds.
–	Practice Test 2
–	May 1st
1	Let ABC be a triangle in which $CA > BC > AB$. Let H be its orthocentre and O its circumcentre. Let D and E be respectively the midpoints of the arc AB not containing C and arc AC not containing B. Let D' and E' be respectively the reflections of D in AB and E in AC. Prove that O, H, D', E' lie on a circle if and only if A, D', E' are collinear.
2	For a composite number n, let d_n denote its largest proper divisor. Show that there are infinitely many n for which $d_n + d_{n+1}$ is a perfect square.
3	Every cell of a 3×3 board is coloured either by red or blue. Find the number of all colorings in which there are no 2×2 squares in which all cells are red.
–	TST 1
–	May 6th
1	Let $ABCD$ be a convex quadrilateral and let the diagonals AC and BD intersect at O. Let I_1, I_2, I_3, I_4 be respectively the incentres of triangles AOB, BOC, COD, DOA. Let J_1, J_2, J_3, J_4 be respectively the excentres of triangles AOB, BOC, COD, DOA opposite O. Show that I_1, I_2, I_3, I_4 lie on a circle if and only if J_1, J_2, J_3, J_4 lie on a circle.

2 Let f and g be two polynomials with integer coefficients such that the leading coefficients of both the polynomials are positive. Suppose $\deg(f)$ is odd and the sets $\{f(a) \mid a \in \mathbb{Z}\}$ and $\{g(a) \mid a \in \mathbb{Z}\}$ are the same. Prove that there exists an integer k such that $g(x) = f(x + k)$.

---

3 Let n points be given inside a rectangle R such that no two of them lie on a line parallel to one of the sides of R. The rectangle R is to be dissected into smaller rectangles with sides parallel to the sides of R in such a way that none of these rectangles contains any of the given points in its interior. Prove that we have to dissect R into at least $n + 1$ smaller rectangles.

Proposed by Serbia

– TST 2

– May 7th

---

1 Let $n \geq 2$ be an integer, and let A_n be the set

$$A_n = \{2^n - 2^k \mid k \in \mathbb{Z},\, 0 \leq k < n\}.$$

Determine the largest positive integer that cannot be written as the sum of one or more (not necessarily distinct) elements of A_n .

Proposed by Serbia

---

2 Find all functions from $\mathbb{N} \cup \{0\} \rightarrow \mathbb{N} \cup \{0\}$ such that $f(m^2 + mf(n)) = mf(m + n)$, for all $m, n \in \mathbb{N} \cup \{0\}$.

---

3 Let G be a simple graph on the infinite vertex set $V = \{v_1, v_2, v_3, \ldots\}$. Suppose every subgraph of G on a finite vertex subset is 10-colorable, Prove that G itself is 10-colorable.

– TST 3

– May 13th

---

1 In a triangle ABC, a point D is on the segment BC, Let X and Y be the incentres of triangles ACD and ABD respectively. The lines BY and CX intersect the circumcircle of triangle AXY at $P \neq Y$ and $Q \neq X$, respectively. Let K be the point of intersection of lines PX and QY. Suppose K is also the reflection of I in BC where I is the incentre of triangle ABC. Prove that $\angle BAC = \angle ADC = 90^\circ$.

---

2 Find all triples (p, x, y) consisting of a prime number p and two positive integers x and y such that $x^{p-1} + y$ and $x + y^{p-1}$ are both powers of p.

Proposed by Belgium

---

3 There are $n \geq 2$ lamps, each with two states: **on** or **off**. For each non-empty subset A of the set of these lamps, there is a *soft-button* which operates on the lamps in A; that is, upon *operating* this button each of the lamps in A changes its state(on to off and off to on). The buttons are identical and it is not known which button corresponds to which subset of lamps. Suppose all the lamps are off initially. Show that one can always switch all the lamps on by performing at most $2^{n-1}+1$ operations.

– TST 4

– May 14th

1 Consider a fixed circle Γ with three fixed points $A, B,$ and C on it. Also, let us fix a real number $\lambda \in (0,1)$. For a variable point $P \notin \{A, B, C\}$ on Γ, let M be the point on the segment CP such that $CM = \lambda \cdot CP$. Let Q be the second point of intersection of the circumcircles of the triangles AMP and BMC. Prove that as P varies, the point Q lies on a fixed circle.

Proposed by Jack Edward Smith, UK

2 Let A be a finite set of pairs of real numbers such that for any pairs (a,b) in A we have $a>0$. Let $X_0=(x_0,y_0)$ be a pair of real numbers(not necessarily from A). We define $X_{j+1}=(x_{j+1},y_{j+1})$ for all $j\geq 0$ as follows: for all $(a,b)\in A$, if $ax_j+by_j>0$ we let $X_{j+1}=X_j$; otherwise we choose a pair (a,b) in A for which $ax_j+by_j\leq 0$ and set $X_{j+1}=(x_j+a,y_j+b)$. Show that there exists an integer $N\geq 0$ such that $X_{N+1}=X_N$.

3 Let $n>1$ be a given integer. Prove that infinitely many terms of the sequence $(a_k)_{k\ge 1}$, defined by

$$a_k=\left\lfloor\frac{n^k}{k}\right\rfloor,$$

are odd. (For a real number x, $\lfloor x\rfloor$ denotes the largest integer not exceeding x.)

Proposed by Hong Kong