14th Middle European Mathematical Olympiad 2020

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Let $\mathbb{N}$ be the set of positive integers. Determine all positive integers k for which there exist functions $f:\mathbb{N}\to\mathbb{N}$ and $g:\mathbb{N}\to\mathbb{N}$ such that g assumes infinitely many values and such that

$$f^{g(n)}(n)=f(n)+k$$

holds for every positive integer n.

(*Remark*. Here, f^i denotes the function f applied i times i.e $f^i(j)=f(f(\dots f(j)\dots))$.)

We call a positive integer N *contagious* if there are 1000 consecutive non-negative integers such that the sum of all their digits is N. Find all contagious positive integers.

Let ABC be an acute scalene triangle with circumcircle ω and incenter I. Suppose the orthocenter H of BIC lies inside ω. Let M be the midpoint of the longer arc BC of ω. Let N be the midpoint of the shorter arc AM of ω.
Prove that there exists a circle tangent to ω at N and tangent to the circumcircles of BHI and CHI.

Find all positive integers n for which there exist positive integers $x_1,x_2,\dots,x_n$ such that

$$\frac{1}{x_1^2}+\frac{2}{x_2^2}+\frac{2^2}{x_3^2}+\cdots+\frac{2^{n-1}}{x_n^2}=1.$$
