The problems from the 27th Macedonian National Olympiad

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1 Let a, b be positive integers and p, q be prime numbers for which $p \nmid q-1$ and $q \mid a^p - b^p$. Prove that $q \mid a-b$.

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2 Let $x_1, ..., x_n$ ($n \geq 2$) be real numbers from the interval $[1, 2]$. Prove that

$|x_1 - x_2| + ... + |x_n - x_1| \leq \frac{2}{3}(x_1 + ... + x_n)$,

with equality holding if and only if n is even and the n-tuple $(x_1, x_2, ..., x_{n-1}, x_n)$ is equal to $(1, 2, ..., 1, 2)$ or $(2, 1, ..., 2, 1)$.

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3 Let ABC be a triangle, and A_1, B_1, C_1 be points on the sides BC, CA, AB, respectively, such that AA_1, BB_1, CC_1 are the internal angle bisectors of $\triangle ABC$. The circumcircle $k' = (A_1B_1C_1)$ touches the side BC at A_1. Let B_2 and C_2, respectively, be the second intersection points of k' with lines AC and AB. Prove that $|AB| = |AC|$ or $|AC_1| = |AB_2|$.

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4 Let S be a nonempty finite set, and $\mathcal{F}$ be a collection of subsets of S such that the following conditions are met:
(i) $\mathcal{F} \setminus S \neq \emptyset$;
(ii) if $F_1, F_2 \in \mathcal{F}$, then $F_1 \cap F_2 \in \mathcal{F}$ and $F_1 \cup F_2 \in \mathcal{F}$.

Prove that there exists $a \in S$ which belongs to at most half of the elements of $\mathcal{F}$.
