JBMO Shortlist 2019

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–	Algebra
A1	Real numbers a and b satisfy $a^3+b^3-6ab=-11$. Prove that $-\frac{7}{3}<a+b<-2$. *Proposed by Serbia*
A2	Let a,b,c be positive real numbers such that $abc=\frac{2}{3}$. Prove that: $$\frac{ab}{a+b}+\frac{bc}{b+c}+\frac{ca}{c+a}\geqslant\frac{a+b+c}{a^3+b^3+c^3}.$$
A3	Let A and B be two non-empty subsets of $X=\{1,2,...,11\}$ with $A\cup B=X$. Let P_A be the product of all elements of A and let P_B be the product of all elements of B. Find the minimum and maximum possible value of P_A+P_B and find all possible equality cases. *Proposed by Greece*
A4	Let a, b be two distinct real numbers and let c be a positive real numbers such that $a^4-2019a=b^4-2019b=c$. Prove that $-\sqrt{c}<ab<0$.
A5	Let a,b,c,d be positive real numbers such that $abcd=1$. Prove the inequality $\frac{1}{a^3+b+c+d}+\frac{1}{a+b^3+c+d}+\frac{1}{a+b+c^3+d}+\frac{1}{a+b+c+d^3}\leq\frac{a+b+c+d}{4}$ *Proposed by Romania*
A6	Let a,b,c be positive real numbers. Prove the inequality $(a^2+ac+c^2)\left(\frac{1}{a+b+c}+\frac{1}{a+c}\right)+b^2\left(\frac{1}{b+c}+\frac{1}{a+b}\right)>a+b+c$. *Proposed by Tajikistan*
A7	Show that for any positive real numbers a,b,c such that $a+b+c=ab+bc+ca$, the following inequality holds $3+\sqrt[3]{\frac{a^3+1}{2}}+\sqrt[3]{\frac{b^3+1}{2}}+\sqrt[3]{\frac{c^3+1}{2}}\leq 2(a+b+c)$ *Proposed by Dorlir Ahmeti, Albania*

– Combinatorics

C1 Let S be a set of 100 positive integer numbers having the following property:
”Among every four numbers of S, there is a number which divides each of the other three or there is a number which is equal to the sum of the other three.”
Prove that the set S contains a number which divides all other 99 numbers of S.

Proposed by Tajikistan

C2 In a certain city there are n straight streets, such that every two streets intersect, and no three streets pass through the same intersection. The City Council wants to organize the city by designating the main and the side street on every intersection. Prove that this can be done in such way that if one goes along one of the streets, from its beginning to its end, the intersections where this street is the main street, and the ones where it is not, will apear in alternating order.

Proposed by Serbia

C3 A 5×100 table is divided into 500 unit square cells, where n of them are coloured black and the rest are coloured white. Two unit square cells are called *adjacent* if they share a common side. Each of the unit square cells has at most two adjacent black unit square cells. Find the largest possible value of n.

C4 We have a group of n kids. For each pair of kids, at least one has sent a message to the other one. For each kid A, among the kids to whom A has sent a message, exactly 25% have sent a message to A. How many possible two-digit values of n are there?

Proposed by Bulgaria

C5 An economist and a statistician play a game on a calculator which does only one operation. The calculator displays only positive integers and it is used in the following way: Denote by n an integer that is shown on the calculator. A person types an integer, m, chosen from the set $\{1, 2, ..., 99\}$ of the first 99 positive integers, and if $m\%$ of the number n is again a positive integer, then the calculator displays $m\%$ of n. Otherwise, the calculator shows an error message and this operation is not allowed. The game consists of doing alternatively these operations and the player that cannot do the operation looses. How many numbers from $\{1, 2, ..., 2019\}$ guarantee the winning strategy for the statistician, who plays second?

For example, if the calculator displays 1200, the economist can type 50, giving the number 600 on the calculator, then the statistician can type 25 giving the number 150. Now, for instance, the economist cannot type 75 as 75% of 150 is not a positive integer, but can choose 40 and the game continues until one of them cannot type an allowed number

Proposed by Serbia

–	Geometry
G1	Let ABC be a right-angled triangle with $\angle A = 90^\circ$ and $\angle B = 30^\circ$. The perpendicular at the midpoint M of BC meets the bisector BK of the angle B at the point E. The perpendicular bisector of EK meets AB at D. Prove that KD is perpendicular to DE. *Proposed by Greece*
G2	Let ABC be a triangle with circumcircle ω. Let l_B and l_C be two lines through the points B and C, respectively, such that $l_B \parallel l_C$. The second intersections of l_B and l_C with ω are D and E, respectively. Assume that D and E are on the same side of BC as A. Let DA intersect l_C at F and let EA intersect l_B at G. If O, O_1 and O_2 are circumcenters of the triangles ABC, ADG and AEF, respectively, and P is the circumcenter of the triangle OO_1O_2, prove that $l_B \parallel OP \parallel l_C$. *Proposed by Stefan Lozanovski, Macedonia*
G3	Let ABC be a triangle with incenter I. The points D and E lie on the segments CA and BC respectively, such that $CD = CE$. Let F be a point on the segment CD. Prove that the quadrilateral $ABEF$ is circumscribable if and only if the quadrilateral $DIEF$ is cyclic. *Proposed by Dorlir Ahmeti, Albania*
G4	Triangle ABC is such that $AB < AC$. The perpendicular bisector of side BC intersects lines AB and AC at points P and Q, respectively. Let H be the orthocentre of triangle ABC, and let M and N be the midpoints of segments BC and PQ, respectively. Prove that lines HM and AN meet on the circumcircle of ABC.
G5	Let P be a point in the interior of a triangle ABC. The lines AP, BP and CP intersect again the circumcircles of the triangles PBC, PCA and PAB at D, E and F respectively. Prove that P is the orthocenter of the triangle DEF if and only if P is the incenter of the triangle ABC. *Proposed by Romania*
G6	Let ABC be a non-isosceles triangle with incenter I. Let D be a point on the segment BC such that the circumcircle of BID intersects the segment AB at $E \neq B$, and the circumcircle of CID intersects the segment AC at $F \neq C$. The circumcircle of DEF intersects AB and AC at the second points M and N respectively. Let P be the point of intersection of IB and DE, and let Q be the point of intersection of IC and DF. Prove that the three lines EN, FM and PQ are parallel. *Proposed by Saudi Arabia*

G7 Let ABC be a right-angled triangle with $\angle A = 90^{\circ}$. Let K be the midpoint of BC, and let $AKLM$ be a parallelogram with centre C. Let T be the intersection of the line AC and the perpendicular bisector of BM. Let ω_1 be the circle with centre C and radius CA and let ω_2 be the circle with centre T and radius TB. Prove that one of the points of intersection of ω_1 and ω_2 is on the line LM.

Proposed by Greece

– Number Theory

N1 Find all prime numbers p for which there exist positive integers x, y, and z such that the number

$x^p + y^p + z^p - x - y - z$

is a product of exactly three distinct prime numbers.

N2 Find all triples (p, q, r) of prime numbers such that all of the following numbers are integers $\frac{p^2+2q}{q+r}, \frac{q^2+9r}{r+p}, \frac{r^2+3p}{p+q}$

Proposed by Tajikistan

N3 Find all prime numbers p and nonnegative integers $x \neq y$ such that $x^4 - y^4 = p(x^3 - y^3)$.

Proposed by Bulgaria

N4 Find all integers x, y such that $x^3(y+1) + y^3(x+1) = 19$.

Proposed by Bulgaria

N5 Find all positive integers x, y, z such that $45^x - 6^y = 2019^z$

Proposed by Dorlir Ahmeti, Albania

N6 a, b, c are non-negative integers.
Solve: $a! + 5^b = 7^c$

Proposed by Serbia

N7 Find all perfect squares n such that if the positive integer $a \geq 15$ is some divisor n then $a + 15$ is a prime power.

Proposed by Saudi Arabia