China Second Round Olympiad 2020

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1 In triangle ABC, $AB = BC$, and let I be the incentre of $\triangle ABC$. M is the midpoint of segment BI. P lies on segment AC, such that $AP = 3PC$. H lies on line PI, such that $MH \perp PH$. Q is the midpoint of the arc AB of the circumcircle of $\triangle ABC$. Prove that $BH \perp QH$.

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2 Let $n \geq 3$ be a given integer, and let $a_1, a_2, \cdots, a_{2n}, b_1, b_2, \cdots, b_{2n}$ be $4n$ nonnegative reals, such that

$$a_1 + a_2 + \cdots + a_{2n} = b_1 + b_2 + \cdots + b_{2n} > 0,$$

and for any $i = 1, 2, \cdots, 2n$, $a_i a_{i+2} \geq b_i + b_{i+1}$, where $a_{2n+1} = a_1$, $a_{2n+2} = a_2$, $b_{2n+1} = b_1$. Detemine the minimum of $a_1 + a_2 + \cdots + a_{2n}$.

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3 Let $a_1 = 1$, $a_2 = 2$, $a_n = 2a_{n-1} + a_{n-2}$, $n = 3, 4, \cdots$. Prove that for any integer $n \geq 5$, a_n has at least one prime factor p, such that $p \equiv 1 \pmod 4$.

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4 Given a convex polygon with 20 vertexes, there are many ways of traingulation it (as 18 triangles). We call the diagram of triangulation, meaning the 20 vertexes, with 37 edges(17 triangluation edges and the original 20 edges), a T-diagram. And the subset of this T-diagram with 10 edges which covers all 20 vertexes(meaning any two edges in the subset doesn't cover the same vertex) calls a ”perfect matching” of this T-diagram. Among all the T-diagrams, find the maximum number of ”perfect matching” of a T-diagram.
