NIMO Summer Contest 2015

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by hwl0304, Binomial-theorem

– August 8th

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1 For all real numbers a and b, let
$$a \bowtie b = \frac{a+b}{a-b}.$$
Compute $1008 \bowtie 1007$.
Proposed by David Altizio

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2 On a 30 question test, Question 1 is worth one point, Question 2 is worth two points, and so on up to Question 30. David takes the test and afterward finds out he answered nine of the questions incorrectly. However, he was not told which nine were incorrect. What is the highest possible score he could have attained?
Proposed by David Altizio

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3 A list of integers with average 89 is split into two disjoint groups. The average of the integers in the first group is 73 while the average of the integers in the second group is 111. What is the smallest possible number of integers in the original list?
Proposed by David Altizio

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4 Let P be a function defined by $P(t) = a^t + b^t$, where a and b are complex numbers. If $P(1) = 7$ and $P(3) = 28$, compute $P(2)$.

Proposed by Justin Stevens

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5 Let $\triangle ABC$ have $AB = 3$, $AC = 5$, and $\angle A = 90^\circ$. Point D is the foot of the altitude from A to $\overline{BC}$, and X and Y are the feet of the altitudes from D to $\overline{AB}$ and $\overline{AC}$ respectively. If XY^2 can be written in the form $\frac{m}{n}$ where m and n are positive relatively prime integers, what is $100m+n$?

Proposed by David Altizio

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6 Let $S_0 = \varnothing$ denote the empty set, and define $S_n = \{S_0, S_1, \dots, S_{n-1}\}$ for every positive integer n. Find the number of elements in the set
$$(S_{10} \cap S_{20}) \cup (S_{30} \cap S_{40}).$$

Proposed by Evan Chen

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7 The NIMO problem writers have invented a new chess piece called the *Oriented Knight*. This new chess piece has a limited number of moves: it can either move two squares to the right and one square upward or two squares upward and one square to the right. How many ways can the knight move from the bottom-left square to the top-right square of a 16×16 chess board?

Proposed by Tony Kim and David Altizio

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8 It is given that the number $4^{11} + 1$ is divisible by some prime greater than 1000. Determine this prime.

Proposed by David Altizio

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9 On a blackboard lies 50 magnets in a line numbered from 1 to 50, with different magnets containing different numbers. David walks up to the blackboard and rearranges the magnets into some arbitrary order. He then writes underneath each pair of consecutive magnets the positive difference between the numbers on the magnets. If the expected number of times he writes the number 1 can be written in the form $\frac{m}{n}$ for relatively prime positive integers m and n, compute $100m + n$.

Proposed by David Altizio

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10 Let $ABCD$ be a tetrahedron with $AB = CD = 1300$, $BC = AD = 1400$, and $CA = BD = 1500$. Let O and I be the centers of the circumscribed sphere and inscribed sphere of $ABCD$, respectively. Compute the smallest integer greater than the length of OI.

Proposed by Michael Ren

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11 We say positive integer n is *metallic* if there is no prime of the form $m^2 - n$. What is the sum of the three smallest metallic integers?

Proposed by Lewis Chen

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12 Let ABC be a triangle whose angles measure A, B, C, respectively. Suppose $\tan A$, $\tan B$, $\tan C$ form a geometric sequence in that order. If $1 \le \tan A + \tan B + \tan C \le 2015$, find the number of possible integer values for $\tan B$. (The values of $\tan A$ and $\tan C$ need not be integers.)

Proposed by Justin Stevens

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13 Let $\triangle ABC$ be a triangle with $AB = 85$, $BC = 125$, $CA = 140$, and incircle ω. Let D, E, F be the points of tangency of ω with $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ respectively, and furthermore denote by X, Y, and Z the incenters of $\triangle AEF$, $\triangle BFD$, and $\triangle CDE$, also respectively. Find the circumradius of $\triangle XYZ$.

Proposed by David Altizio

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14 We say that an integer a is a quadratic, cubic, or quintic residue modulo n if there exists an integer x such that $x^2 \equiv a \pmod{n}$, $x^3 \equiv a \pmod{n}$, or $x^5 \equiv a \pmod{n}$, respectively. Further, an integer a is a primitive residue modulo n if it is exactly one of these three types of residues modulo n.

How many integers $1 \le a \le 2015$ are primitive residues modulo 2015?

Proposed by Michael Ren

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15 Suppose x and y are real numbers such that

$$x^2 + xy + y^2 = 2 \qquad \text{and} \qquad x^2 - y^2 = \sqrt{5}.$$

The sum of all possible distinct values of $|x|$ can be written in the form $\sum_{i=1}^n \sqrt{a_i}$, where each of the a_i is a rational number. If $\sum_{i=1}^n a_i = \frac{m}{n}$ where m and n are positive realtively prime integers, what is $100m + n$?

Proposed by David Altizio
