

IMO Shortlist 2019

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– Algebra

A1 Let \mathbb{Z} be the set of integers. Determine all functions $f : \mathbb{Z} \rightarrow \mathbb{Z}$ such that, for all integers a and b ,

$$f(2a) + 2f(b) = f(f(a + b)).$$

Proposed by Liam Baker, South Africa

A2 Let $u_1, u_2, \dots, u_{2019}$ be real numbers satisfying

$$u_1 + u_2 + \dots + u_{2019} = 0 \quad \text{and} \quad u_1^2 + u_2^2 + \dots + u_{2019}^2 = 1.$$

Let $a = \min(u_1, u_2, \dots, u_{2019})$ and $b = \max(u_1, u_2, \dots, u_{2019})$. Prove that

$$ab \leq -\frac{1}{2019}.$$

A3 Let $n \geq 3$ be a positive integer and let (a_1, a_2, \dots, a_n) be a strictly increasing sequence of n positive real numbers with sum equal to 2. Let X be a subset of $\{1, 2, \dots, n\}$ such that the value of

$$\left| 1 - \sum_{i \in X} a_i \right|$$

is minimised. Prove that there exists a strictly increasing sequence of n positive real numbers (b_1, b_2, \dots, b_n) with sum equal to 2 such that

$$\sum_{i \in X} b_i = 1.$$

A4 Let $n \geq 2$ be a positive integer and a_1, a_2, \dots, a_n be real numbers such that

$$a_1 + a_2 + \dots + a_n = 0.$$

Define the set A by

$$A = \{(i, j) \mid 1 \leq i < j \leq n, |a_i - a_j| \geq 1\}$$

Prove that, if A is not empty, then

$$\sum_{(i,j) \in A} a_i a_j < 0.$$

A5 Let x_1, x_2, \dots, x_n be different real numbers. Prove that

$$\sum_{1 \leq i \leq n} \prod_{j \neq i} \frac{1 - x_i x_j}{x_i - x_j} = \begin{cases} 0, & \text{if } n \text{ is even;} \\ 1, & \text{if } n \text{ is odd.} \end{cases}$$

A6 A polynomial $P(x, y, z)$ in three variables with real coefficients satisfies the identities

$$P(x, y, z) = P(x, y, xy - z) = P(x, zx - y, z) = P(yz - x, y, z).$$

Prove that there exists a polynomial $F(t)$ in one variable such that

$$P(x, y, z) = F(x^2 + y^2 + z^2 - xyz).$$

A7 Let \mathbb{Z} be the set of integers. We consider functions $f : \mathbb{Z} \rightarrow \mathbb{Z}$ satisfying

$$f(f(x + y) + y) = f(f(x) + y)$$

for all integers x and y . For such a function, we say that an integer v is *f-rare* if the set

$$X_v = \{x \in \mathbb{Z} : f(x) = v\}$$

is finite and nonempty.

(a) Prove that there exists such a function f for which there is an *f-rare* integer.

(b) Prove that no such function f can have more than one *f-rare* integer.

Netherlands

– Combinatorics

C1 The infinite sequence a_0, a_1, a_2, \dots of (not necessarily distinct) integers has the following properties: $0 \leq a_i \leq i$ for all integers $i \geq 0$, and

$$\binom{k}{a_0} + \binom{k}{a_1} + \dots + \binom{k}{a_k} = 2^k$$

for all integers $k \geq 0$. Prove that all integers $N \geq 0$ occur in the sequence (that is, for all $N \geq 0$, there exists $i \geq 0$ with $a_i = N$).

C2 You are given a set of n blocks, each weighing at least 1; their total weight is $2n$. Prove that for every real number r with $0 \leq r \leq 2n - 2$ you can choose a subset of the blocks whose total weight is at least r but at most $r + 2$.

- C3** The Bank of Bath issues coins with an H on one side and a T on the other. Harry has n of these coins arranged in a line from left to right. He repeatedly performs the following operation: if there are exactly $k > 0$ coins showing H , then he turns over the k th coin from the left; otherwise, all coins show T and he stops. For example, if $n = 3$ the process starting with the configuration THT would be $THT \rightarrow HHT \rightarrow HTT \rightarrow TTT$, which stops after three operations.

(a) Show that, for each initial configuration, Harry stops after a finite number of operations.

(b) For each initial configuration C , let $L(C)$ be the number of operations before Harry stops. For example, $L(THT) = 3$ and $L(TTT) = 0$. Determine the average value of $L(C)$ over all 2^n possible initial configurations C .

Proposed by David Altizio, USA

- C4** On a flat plane in Camelot, King Arthur builds a labyrinth \mathcal{L} consisting of n walls, each of which is an infinite straight line. No two walls are parallel, and no three walls have a common point. Merlin then paints one side of each wall entirely red and the other side entirely blue.

At the intersection of two walls there are four corners: two diagonally opposite corners where a red side and a blue side meet, one corner where two red sides meet, and one corner where two blue sides meet. At each such intersection, there is a two-way door connecting the two diagonally opposite corners at which sides of different colours meet.

After Merlin paints the walls, Morgana then places some knights in the labyrinth. The knights can walk through doors, but cannot walk through walls.

Let $k(\mathcal{L})$ be the largest number k such that, no matter how Merlin paints the labyrinth \mathcal{L} , Morgana can always place at least k knights such that no two of them can ever meet. For each n , what are all possible values for $k(\mathcal{L})$, where \mathcal{L} is a labyrinth with n walls?

- C5** A social network has 2019 users, some pairs of whom are friends. Whenever user A is friends with user B , user B is also friends with user A . Events of the following kind may happen repeatedly, one at a time:

- Three users A , B , and C such that A is friends with both B and C , but B and C are not friends, change their friendship statuses such that B and C are now friends, but A is no longer friends with B , and no longer friends with C . All other friendship statuses are unchanged.

Initially, 1010 users have 1009 friends each, and 1009 users have 1010 friends each. Prove that there exists a sequence of such events after which each user is friends with at most one other user.

Proposed by Adrian Beker, Croatia

- C6** Let $n > 1$ be an integer. Suppose we are given $2n$ points in the plane such that no three of them are collinear. The points are to be labelled A_1, A_2, \dots, A_{2n} in some order. We then consider the $2n$ angles $\angle A_1 A_2 A_3, \angle A_2 A_3 A_4, \dots, \angle A_{2n-2} A_{2n-1} A_{2n}, \angle A_{2n-1} A_{2n} A_1, \angle A_{2n} A_1 A_2$. We measure

each angle in the way that gives the smallest positive value (i.e. between 0° and 180°). Prove that there exists an ordering of the given points such that the resulting $2n$ angles can be separated into two groups with the sum of one group of angles equal to the sum of the other group.

- C7** There are 60 empty boxes B_1, \dots, B_{60} in a row on a table and an unlimited supply of pebbles. Given a positive integer n , Alice and Bob play the following game. In the first round, Alice takes n pebbles and distributes them into the 60 boxes as she wishes. Each subsequent round consists of two steps:
- Bob chooses an integer k with $1 \leq k \leq 59$ and splits the boxes into the two groups B_1, \dots, B_k and B_{k+1}, \dots, B_{60} .
 - Alice picks one of these two groups, adds one pebble to each box in that group, and removes one pebble from each box in the other group.
- Bob wins if, at the end of any round, some box contains no pebbles. Find the smallest n such that Alice can prevent Bob from winning.

Czech Republic

- C8** Alice has a map of Wonderland, a country consisting of $n \geq 2$ towns. For every pair of towns, there is a narrow road going from one town to the other. One day, all the roads are declared to be "one way" only. Alice has no information on the direction of the roads, but the King of Hearts has offered to help her. She is allowed to ask him a number of questions. For each question in turn, Alice chooses a pair of towns and the King of Hearts tells her the direction of the road connecting those two towns.

Alice wants to know whether there is at least one town in Wonderland with at most one outgoing road. Prove that she can always find out by asking at most $4n$ questions.

- C9** For any two different real numbers x and y , we define $D(x, y)$ to be the unique integer d satisfying $2^d \leq |x - y| < 2^{d+1}$. Given a set of reals \mathcal{F} , and an element $x \in \mathcal{F}$, we say that the *scales* of x in \mathcal{F} are the values of $D(x, y)$ for $y \in \mathcal{F}$ with $x \neq y$. Let k be a given positive integer.

Suppose that each member x of \mathcal{F} has at most k different scales in \mathcal{F} (note that these scales may depend on x). What is the maximum possible size of \mathcal{F} ?

– Geometry

- G1** Let ABC be a triangle. Circle Γ passes through A , meets segments AB and AC again at points D and E respectively, and intersects segment BC at F and G such that F lies between B and G . The tangent to circle BDF at F and the tangent to circle CEG at G meet at point T . Suppose that points A and T are distinct. Prove that line AT is parallel to BC .

(Nigeria)

- G2** Let ABC be an acute-angled triangle and let D, E , and F be the feet of altitudes from A, B , and

C to sides BC , CA , and AB , respectively. Denote by ω_B and ω_C the incircles of triangles BDF and CDE , and let these circles be tangent to segments DF and DE at M and N , respectively. Let line MN meet circles ω_B and ω_C again at $P \neq M$ and $Q \neq N$, respectively. Prove that $MP = NQ$.

(Vietnam)

- G3** In triangle ABC , point A_1 lies on side BC and point B_1 lies on side AC . Let P and Q be points on segments AA_1 and BB_1 , respectively, such that PQ is parallel to AB . Let P_1 be a point on line PB_1 , such that B_1 lies strictly between P and P_1 , and $\angle PP_1C = \angle BAC$. Similarly, let Q_1 be the point on line QA_1 , such that A_1 lies strictly between Q and Q_1 , and $\angle CQ_1Q = \angle CBA$.

Prove that points P, Q, P_1 , and Q_1 are concyclic.

Proposed by Anton Trygub, Ukraine

- G4** Let P be a point inside triangle ABC . Let AP meet BC at A_1 , let BP meet CA at B_1 , and let CP meet AB at C_1 . Let A_2 be the point such that A_1 is the midpoint of PA_2 , let B_2 be the point such that B_1 is the midpoint of PB_2 , and let C_2 be the point such that C_1 is the midpoint of PC_2 . Prove that points A_2, B_2 , and C_2 cannot all lie strictly inside the circumcircle of triangle ABC .

(Australia)

- G5** Let $ABCDE$ be a convex pentagon with $CD = DE$ and $\angle EDC \neq 2 \cdot \angle ADB$. Suppose that a point P is located in the interior of the pentagon such that $AP = AE$ and $BP = BC$. Prove that P lies on the diagonal CE if and only if $\text{area}(BCD) + \text{area}(ADE) = \text{area}(ABD) + \text{area}(ABP)$.

(Hungary)

- G6** Let I be the incentre of acute-angled triangle ABC . Let the incircle meet BC, CA , and AB at D, E , and F , respectively. Let line EF intersect the circumcircle of the triangle at P and Q , such that F lies between E and P . Prove that $\angle DPA + \angle AQD = \angle QIP$.

(Slovakia)

- G7** Let I be the incentre of acute triangle ABC with $AB \neq AC$. The incircle ω of ABC is tangent to sides BC, CA , and AB at D, E , and F , respectively. The line through D perpendicular to EF meets ω at R . Line AR meets ω again at P . The circumcircles of triangle PCE and PBF meet again at Q .

Prove that lines DI and PQ meet on the line through A perpendicular to AI .

Proposed by Anant Mudgal, India

- G8** Let \mathcal{L} be the set of all lines in the plane and let f be a function that assigns to each line $\ell \in \mathcal{L}$ a point $f(\ell)$ on ℓ . Suppose that for any point X , and for any three lines ℓ_1, ℓ_2, ℓ_3 passing through X , the points $f(\ell_1), f(\ell_2), f(\ell_3)$, and X lie on a circle. Prove that there is a unique point P such that $f(\ell) = P$ for any line ℓ passing through P .

Australia

– Number Theory

- N1** Find all pairs (k, n) of positive integers such that

$$k! = (2^n - 1)(2^n - 2)(2^n - 4) \cdots (2^n - 2^{n-1}).$$

Proposed by Gabriel Chicas Reyes, El Salvador

- N2** Find all triples (a, b, c) of positive integers such that $a^3 + b^3 + c^3 = (abc)^2$.
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- N3** We say that a set S of integers is *rootiful* if, for any positive integer n and any $a_0, a_1, \dots, a_n \in S$, all integer roots of the polynomial $a_0 + a_1x + \cdots + a_nx^n$ are also in S . Find all rootiful sets of integers that contain all numbers of the form $2^a - 2^b$ for positive integers a and b .
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- N4** Find all functions $f : \mathbb{Z}_{>0} \rightarrow \mathbb{Z}_{>0}$ such that $a + f(b)$ divides $a^2 + bf(a)$ for all positive integers a and b with $a + b > 2019$.
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- N5** Let a be a positive integer. We say that a positive integer b is $[i]a$ -good $[/i]$ if $\binom{an}{b} - 1$ is divisible by $an + 1$ for all positive integers n with $an \geq b$. Suppose b is a positive integer such that b is a -good, but $b + 2$ is not a -good. Prove that $b + 1$ is prime.
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- N6** Let $H = \{[i\sqrt{2}] : i \in \mathbb{Z}_{>0}\} = \{1, 2, 4, 5, 7, \dots\}$ and let n be a positive integer. Prove that there exists a constant C such that, if $A \subseteq \{1, 2, \dots, n\}$ satisfies $|A| \geq C\sqrt{n}$, then there exist $a, b \in A$ such that $a - b \in H$. (Here $\mathbb{Z}_{>0}$ is the set of positive integers, and $[z]$ denotes the greatest integer less than or equal to z .)
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- N7** Prove that there is a constant $c > 0$ and infinitely many positive integers n with the following property: there are infinitely many positive integers that cannot be expressed as the sum of fewer than $cn \log(n)$ pairwise coprime n th powers.

Canada

- N8** Let a and b be two positive integers. Prove that the integer

$$a^2 + \left\lceil \frac{4a^2}{b} \right\rceil$$

is not a square. (Here $\lceil z \rceil$ denotes the least integer greater than or equal to z .)

Russia
