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by parmenides51

**–** paper 1

**1** De fine the *quasi-primes* as follows. • The first quasi-prime is $q_1 = 2$ • For $n \geq 2$, the $n^{th}$ quasi-prime $q_n$ is the smallest integer greater than $q_{n_1}$ and not of the form $q_i q_j$ for some $1 \leq i \leq j \leq n-1$.
Determine, with proof, whether or not $1000$ is a quasi-prime.

**2** Jenny is going to attend a sports camp for $7$ days. Each day, she will play exactly one of three sports: hockey, tennis or camogie. The only restriction is that in any period of $4$ consecutive days, she must play all three sports. Find, with proof, the number of possible sports schedules for Jennys week.

**3** A quadrilateral $ABCD$ is such that the sides $AB$ and $DC$ are parallel, and $|BC| = |AB| + |CD|$. Prove that the angle bisectors of the angles $\angle ABC$ and $\angle BCD$ intersect at right angles on the side $AD$.

**4** Find the set of all quadruplets $(x, y, z, w)$ of non-zero real numbers which satisfy

$$1 + \frac{1}{x} + \frac{2(x+1)}{xy} + \frac{3(x+1)(y+2)}{xyz} + \frac{4(x+1)(y+2)(z+3)}{xyzw} = 0$$

**5** Let $M$ be a point on the side $BC$ of triangle $ABC$ and let $P$ and $Q$ denote the circumcentres of triangles $ABM$ and $ACM$ respectively. Let $L$ denote the point of intersection of the extended lines $BP$ and $CQ$ and let $K$ denote the reflection of $L$ through the line $PQ$. Prove that $M, P, Q$ and $K$ all lie on the same circle.

**–** paper 2

**6** The number $2019$ has the following nice properties:
(a) It is the sum of the fourth powers of fuve distinct positive integers.
(b) It is the sum of six consecutive positive integers.
In fact, $2019 = 1^4 + 2^4 + 3^4 + 5^4 + 6^4$ (1) $2019 = 334 + 335 + 336 + 337 + 338 + 339$ (2)
Prove that $2019$ is the smallest number that satis es **both** (a) and (b).
(You may assume that (1) and (2) are correct!)

**7** Three non-zero real numbers $a, b, c$ satisfy $a + b + c = 0$ and $a^4 + b^4 + c^4 = 128$. Determine all possible values of $ab + bc + ca$.

**8** Consider a point $G$ in the interior of a parallelogram $ABCD$. A circle $\Gamma$ through $A$ and $G$ intersects the sides $AB$ and $AD$ for the second time at the points $E$ and $F$ respectively. The line $FG$ extended intersects the side $BC$ at $H$ and the line $EG$ extended intersects the side $CD$ at $I$. The circumcircle of triangle $HGI$ intersects the circle $\Gamma$ for the second time at $M \neq G$. Prove that $M$ lies on the diagonal $AC$.

**9** Suppose $x, y, z$ are real numbers such that $x^2 + y^2 + z^2 + 2xyz = 1$. Prove that $8xyz \leq 1$, with equality if and only if $(x, y, z)$ is one of the following:

$$\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right), \left(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right), \left(-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right), \left(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right)$$

**10** Island Hopping Holidays offer short holidays to $64$ islands, labeled Island $i, 1 \leq i \leq 64$. A guest chooses any Island $a$ for the fi rst night of the holiday, moves to Island $b$ for the second night, and finally moves to Island $c$ for the third night. Due to the limited number of boats, we must have $b \in T_a$ and $c \in T_b$, where the sets $T_i$ are chosen so that
(a) each $T_i$ is non-empty, and $i \notin T_i$,
(b) $\sum_{i=1}^{64} |T_i| = 128$, where $|T_i|$ is the number of elements of $T_i$.
Exhibit a choice of sets $T_i$ giving at least $63 \cdot 64 + 6 = 4038$ possible holidays.
Note that c = a is allowed, and holiday choices $(a, b, c)$ and $(a', b', c')$ are considered distinct if $a \neq a'$ or $b \neq b'$ or $c \neq c'$.