

Tuymaada Olympiad 2020

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– Juniors

– Day 1

1 For each positive integer m let t_m be the smallest positive integer not dividing m . Prove that there are infinitely many positive integers which can not be represented in the form $m + t_m$.

(A. Golovanov)

2 All non-zero coefficients of the polynomial $f(x)$ equal 1, while the sum of the coefficients is 20. Is it possible that thirteen coefficients of $f^2(x)$ equal 9?

(S. Ivanov, K. Kokhas)

3 Each edge of a complete graph with 101 vertices is marked with 1 or -1 . It is known that absolute value of the sum of numbers on all the edges is less than 150. Prove that the graph contains a path visiting each vertex exactly once such that the sum of numbers on all edges of this path is zero.

(Y. Caro, A. Hansberg, J. Lauri, C. Zarb)

4 Points D and E lie on the lines BC and AC respectively so that B is between C and D , C is between A and E , $BC = BD$ and $\angle BAD = \angle CDE$. It is known that the ratio of the perimeters of the triangles ABC and ADE is 2. Find the ratio of the areas of these triangles.

– Day 2

5 Coordinate axes (without any marks, with the same scale) and the graph of a quadratic trinomial $y = x^2 + ax + b$ are drawn in the plane. The numbers a and b are not known. How to draw a unit segment using only ruler and compass?

6 AK and BL are altitudes of an acute triangle ABC . Point P is chosen on the segment AK so that $LK = LP$. The parallel to BC through P meets the parallel to PL through B at point Q . Prove that $\angle AQB = \angle ACB$.

(S. Berlov)

7 How many positive integers N in the segment $[10, 10^{20}]$ are such that if all their digits are increased by 1 and then multiplied, the result is $N + 1$?

(F. Bakharev)

- 8** In a horizontal strip $1 \times n$ made of n unit squares the vertices of all squares are marked. The strip is partitioned into parts by segments connecting marked points and not lying on the sides of the strip. The segments can not have common inner points; the upper end of each segment must be either above the lower end or further to the right. Prove that the number of all partitions is divisible by 2^n . (The partition where no segments are drawn, is counted too.)

(E. Robeva, M. Sun)

– Seniors

– Day 1

- 1** Does the system of equation

$$\begin{cases} x_1 + x_2 = y_1 + y_2 + y_3 + y_4 \\ x_1^2 + x_2^2 = y_1^2 + y_2^2 + y_3^2 + y_4^2 \\ x_1^3 + x_2^3 = y_1^3 + y_2^3 + y_3^3 + y_4^3 \end{cases}$$

admit a solution in integers such that the absolute value of each of these integers is greater than 2020?

- 2** Given positive real numbers a_1, a_2, \dots, a_n . Let

$$m = \min \left(a_1 + \frac{1}{a_2}, a_2 + \frac{1}{a_3}, \dots, a_{n-1} + \frac{1}{a_n}, a_n + \frac{1}{a_1} \right).$$

Prove the inequality

$$\sqrt[n]{a_1 a_2 \dots a_n} + \frac{1}{\sqrt[n]{a_1 a_2 \dots a_n}} \geq m.$$

- 3** same as juniors Q4

- 4** For each positive integer k , let $g(k)$ be the maximum possible number of points in the plane such that pairwise distances between these points have only k different values. Prove that there exists k such that $g(k) > 2k + 2020$.

– Day 2

- 5** same as juniors Q5

6 An isosceles triangle ABC ($AB = BC$) is given. Circles ω_1 and ω_2 with centres O_1 and O_2 lie in the angle ABC and touch the sides AB and CB at A and C respectively, and touch each other externally at point X . The side AC meets the circles again at points Y and Z . O is the circumcenter of the triangle XYZ . Lines O_2O and O_1O intersect lines AB and BC at points C_1 and A_1 respectively. Prove that B is the circumcentre of the triangle A_1OC_1 .

7 Several policemen try to catch a thief who has $2m$ accomplices. To that end they place the accomplices under surveillance. In the beginning, the policemen shadow nobody. Every morning each policeman places under his surveillance one of the accomplices. Every evening the thief stops trusting one of his accomplices. The thief is caught if by the m -th evening some policeman shadows exactly those m accomplices who are still trusted by the thief. Prove that to guarantee the capture of the thief at least 2^m policemen are needed.

8 The degrees of polynomials P and Q with real coefficients do not exceed n . These polynomials satisfy the identity

$$P(x)x^{n+1} + Q(x)(x+1)^{n+1} = 1.$$

Determine all possible values of $Q\left(-\frac{1}{2}\right)$.
