2nd Final Mathematical Cup 2020 FMC

www.artofproblemsolving.com/community/c1332626

by parmenides51, Miku3D, sqing

–	Juniors
1	Let n be a given positive integer. Prove that there is no positive divisor d of $2n^2$ such that $d^2n^2 + d^3$ is a square of an integer.
2	Let a, b, c be positive real numbers . Prove that $$\frac{1}{ab(b+1)(c+1)}+\frac{1}{bc(c+1)(a+1)}+\frac{1}{ca(a+1)(b+1)}\ge\frac{3}{(1+abc)^2}.$$
3	Let k,n be positive integers, $k, n > 1$, $k < n$ and a $n \times n$ grid of unit squares is given. Ana and Maya take turns in coloring the grid in the following way: in each turn, a unit square is colored black in such a way that no two black cells have a common side or vertex. Find the smallest positive integer n , such that they can obtain a configuration in which each row and column contains exactly k black cells. Draw one example.
4	Let ABC be a triangle such that $\measuredangle BAC = 60^\circ$. Let D and E be the feet of the perpendicular from A to the bisectors of the external angles of B and C in triangle ABC, respectively. Let O be the circumcenter of the triangle ABC. Prove that circumcircle of the triangle BOC has exactly one point in common with the circumcircle of ADE.
–	Seniors
1	Find all such functions $f:\mathbb{R}\to\mathbb{R}$ that for any real x, y the following equation is true. $$f(f(x)+y)+1=f(x^2+y)+2f(x)+2y$$
2	same as Junior Q4
3	Given a paper on which the numbers $1, 2, 3 \dots, 14, 15$ are written. Andy and Bobby are bored and perform the following operations, Andy chooses any two numbers (say x and y) on the paper, erases them, and writes the sum of the numbers on the initial paper. Meanwhile, Bobby writes the value of $xy(x+y)$ in his book. They were so bored that they both performed the operation until only 1 number remained. Then Bobby adds up all the numbers he wrote in his book, lets call k as the sum. a. Prove that k is constant which means it does not matter how they perform the operation, b. Find the value of k.

4 Find all positive integers n such that for all positive integers m, $1 < m < n$, relatively prime to n, m must be a prime number.