Czech-Polish-Slovak Match 2020

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by parmenides51

– Day 1

1 Let $ABCD$ be a parallelogram whose diagonals meet at P. Denote by M the midpoint of AB. Let Q be a point such that QA is tangent to the circumcircle of MAD and QB is tangent to the circumcircle of MBC. Prove that points Q, M, P are collinear.

(Patrik Bak, Slovakia)

2 Given a positive integer n, we say that a real number x is n-good if there exist n positive integers $a_1, ..., a_n$ such that

$$x = \frac{1}{a_1} + ... + \frac{1}{a_n}.$$

Find all positive integers k for which the following assertion is true:
if a, b are real numbers such that the closed interval $[a, b]$ contains infinitely many 2020-good numbers, then the interval $[a, b]$ contains at least one k-good number.

(Josef Tkadlec, Czech Republic)

3 The numbers $1, 2, ..., 2020$ are written on the blackboard. Venus and Serena play the following game. First, Venus connects by a line segment two numbers such that one of them divides the other. Then Serena connects by a line segment two numbers which has not been connected and such that one of them divides the other. Then Venus again and they continue until there is a triangle with one vertex in 2020, i.e. 2020 is connected to two numbers that are connected with each other. The girl that has drawn the last line segment (completed the triangle) is the winner. Which of the girls has a winning strategy?

(Tomáš Bárta, Czech Republic)

– Day 2

4 Let a be a given real number. Find all functions $f : R \to R$ such that $(x+y)(f(x)-f(y)) = a(x-y)f(x+y)$ holds for all $x, y \in R$.

(Walther Janous, Austria)

5 Let n be a positive integer and let $d(n)$ denote the number of ordered pairs of positive integers (x, y) such that $(x+1)^2 - xy(2x-xy+2y) + (y+1)^2 = n$. Find the smallest positive integer n satisfying $d(n) = 61$.

(Patrik Bak, Slovakia)

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6 Let ABC be an acute triangle. Let P be a point such that PB and PC are tangent to circumcircle of ABC. Let X and Y be variable points on AB and AC, respectively, such that $\angle XPY = 2\angle BAC$ and P lies in the interior of triangle AXY. Let Z be the reflection of A across XY. Prove that the circumcircle of XYZ passes through a fixed point.

(Dominik Burek, Poland)
