239 Open Mathematical Olympiad 2016

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– Grade 10-11

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2 In triangle ABC, the incircle touches sides AB and BC at points P and Q, respectively. Median of triangle ABC from vertex B meets segment PQ at point R. Prove that angle ARC is obtuse.

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3 Positive real numbers a,b,c are given such that $abc=1$.Prove that

$$2(a+b+c)+\frac{9}{(ab+bc+ca)^2}\geq 7.$$

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4 The sequences of natural numbers p_n and q_n are given such that

$$p_1=1,\ q_1=1,\ p_{n+1}=2q_n^2-p_n^2,\ q_{n+1}=2q_n^2+p_n^2$$

Prove that p_n and q_m are coprime for any m and n.

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5 Through point P inside triangle ABC, straight lines were drawn, parallel to the sides, until they intersect with the sides. In the three resulting parallelograms, diagonals that do not contain point P, are drawn. Points A_1, B_1 and C_1 are the intersection points of the lines containing these diagonals such that A_1 and A are in different sides of line BC and B_1 and C_1 are similar. Prove that if hexagon $AC_1BA_1CB_1$ is inscribed and convex, then point P is the orthocenter of triangle $A_1B_1C_1$.

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6 A finite family of finite sets F is given, satisfying two conditions:
(i) if $A, B \in F$, then $A \cup B \in F$;
(ii) if $A \in F$, then the number of elements $|A|$ is not a multiple of 3.
Prove that you can specify at most two elements so that every set of the family F contains at least one of them.

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7 Find all functions $f:\mathbb{R}^+ \to \mathbb{R}^+$ satisfying

$$f(xy+x+y)=(f(x)-f(y))f(y-x-1)$$

for all $x>0, y>x+1$.

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8 Given a natural number $k>1$. Find the smallest number α satisfying the following condition. Suppose that the table $(2k+1)\times(2k+1)$ is filled with real numbers not exceeding 1 in absolute value, and the sums of the numbers in all lines are equal to zero. Then you can rearrange the numbers so that each number remains in its row and all the sums over the columns will be at most α.

– Grade 8-9

1 A natural number $k > 1$ is given. The sum of some divisor of k and some divisor of $k - 1$ is equal to a,where $a > k + 1$. Prove that at least one of the numbers $a - 1$ or $a + 1$ composite.

2 In a convex quadrilateral $ABCD$ rays AB and DC intersect at point P, and rays BC and AD at point Q.
There is a point T on the diagonal AC such that the triangles BTP and DTQ are similar, in that order. Prove that $BD \| PQ$.

3 A regular hexagon with a side of 50 was divided to equilateral triangles with unit side, parallel to the sides of the hexagon. It is allowed to delete any three nodes of the resulting lattice defining a segment of length 2. As a result of several such operations, exactly one node remains. How many ways is this possible?

4 Positive real numbers a, b, c are given such that $abc = 1$. Prove that

$$a + b + c + \frac{3}{ab + bc + ca} \geq 4.$$

5 Triangle ABC in which $AB < BC$, is inscribed in a circle ω and circumscribed about a circle γ with center I. The line ℓ parallel to AC, touches the circle γ and intersects the arcs BAC and BCA at points P and Q, respectively. It is known that $PQ = 2BI$. Prove that $AP + 2PB = CP$.

6 A graph is called $7 - chip$ if it obtained by removing at most three edges that have no vertex in common from a complete graph with seven vertices. Consider a complete graph G with v vertices which each edge of its is colored blue or red. Prove that there is either a blue path with 100 edges or a red $7 - chip$.

7 A set is called $six\ square$ if it has six pair-wise coprime numbers and for any partition of it into two set with three elements, the sum of the numbers in one of them is perfect square. Prove that there exist infinitely many $six\ square$.

8 There are n triangles inscribed in a circle and all $3n$ of their vertices are different. Prove that it is possible to put a boy in one of the vertices in each triangle, and a girl in the other, so that boys and girls alternate on a circle.