National Science Olympiad 2020

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–	Day 1	

1 Since this is already 3 PM (GMT +7) in Jakarta, might as well post the problem here.

Problem 1. Given an acute triangle ABC and the point D on segment BC. Circle c_1 passes through A, D and its centre lies on AC. Whereas circle c_2 passes through A, D and its centre lies on AB. Let $P \neq A$ be the intersection of c_1 with AB and $Q \neq A$ be the intersection of c_2 with AC. Prove that AD bisects $\angle PDQ$.

2 Problem 2. Let $P(x) = ax^2 + bx + c$ where a, b, c are real numbers. If

$$P(a) = bc, \quad P(b) = ac, \quad P(c) = ab$$

then prove that

$$(a-b)(b-c)(c-a)(a+b+c) = 0.$$

3 The wording is just ever so slightly different, however the problem is identical.

Problem 3. Determine all functions $f : \mathbb{N} \to \mathbb{N}$ such that $n^2 + f(n)f(m)$ is a multiple of $f(n) + m$ for all natural numbers m, n.

4 Problem 4. A chessboard with $2n \times 2n$ tiles is coloured such that every tile is coloured with one out of n colours. Prove that there exists 2 tiles in either the same column or row such that if the colours of both tiles are swapped, then there exists a rectangle where all its four corner tiles have the same colour.

– Day 2

5 A set A contains exactly n integers, each of which is greater than 1 and every of their prime factors is less than 10. Determine the smallest n such that A must contain at least two distinct elements a and b such that ab is the square of an integer.

6 Given a cyclic quadrilateral $ABCD$. Let X be a point on segment BC ($X \neq BC$) such that line AX is perpendicular to the angle bisector of $\angle CBD$, and Y be a point on segment AD ($Y \neq D$) such that BY is perpendicular to the angle bisector of $\angle CAD$. Prove that XY is parallel to CD.

7 Determine all real-coefficient polynomials $P(x)$ such that

$$P(\lfloor x \rfloor) = \lfloor P(x) \rfloor$$

for every real numbers x.

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8 Determine the smallest natural number $n > 2$, or show that no such natural numbers n exists, that satisfy the following condition: There exists natural numbers $a_1, a_2, \dots, a_n$ such that

$$\gcd(a_1, a_2, \dots, a_n) = \sum_{k=1}^{n-1} \underbrace{\left(\frac{1}{\gcd(a_k, a_{k+1})} + \frac{1}{\gcd(a_k, a_{k+2})} + \dots + \frac{1}{\gcd(a_k, a_n)} \right)}_{n-k \text{ terms}}$$
