European Mathematical Cup 2012

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by Sayan, Matematika

– Junior Division

1 Let ABC be a triangle and Q a point on the internal angle bisector of $\angle BAC$. Circle ω_1 is circumscribed to triangle BAQ and intersects the segment AC in point $P \neq C$. Circle ω_2 is circumscribed to the triangle CQP. Radius of the cirlce ω_1 is larger than the radius of ω_2. Circle centered at Q with radius QA intersects the circle ω_1 in points A and A_1. Circle centered at Q with radius QC intersects ω_1 in points C_1 and C_2. Prove $\angle A_1BC_1 = \angle C_2PA$.

Proposed by Matija Buci.

2 Let S be the set of positive integers. For any a and b in the set we have $GCD(a,b) > 1$. For any a, b and c in the set we have $GCD(a,b,c) = 1$. Is it possible that S has 2012 elements?

Proposed by Ognjen Stipeti.

3 Are there positive real numbers x, y and z such that

$x^4 + y^4 + z^4 = 13$,

$x^3y^3z + y^3z^3x + z^3x^3y = 6\sqrt{3}$,

$x^3yz + y^3zx + z^3xy = 5\sqrt{3}$?

Proposed by Matko Ljulj.

4 Let k be a positive integer. At the European Chess Cup every pair of players played a game in which somebody won (there were no draws). For any k players there was a player against whom they all lost, and the number of players was the least possible for such k. Is it possible that at the Closing Ceremony all the participants were seated at the round table in such a way that every participant was seated next to both a person he won against and a person he lost against.

Proposed by Matija Buci.

– Senior Division

1 Find all positive integers a, b, n and prime numbers p that satisfy

$$a^{2013} + b^{2013} = p^n.$$

Proposed by Matija Buci.

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2 Let ABC be an acute triangle with orthocenter H. Segments AH and CH intersect segments BC and AB in points A_1 and C_1 respectively. The segments BH and A_1C_1 meet at point D. Let P be the midpoint of the segment BH. Let D' be the reflection of the point D in AC. Prove that quadrilateral $APCD'$ is cyclic.

Proposed by Matko Ljulj.

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3 Prove that the following inequality holds for all positive real numbers a, b, c, d, e and f

$$\sqrt[3]{\frac{abc}{a+b+d}}+\sqrt[3]{\frac{def}{c+e+f}}<\sqrt[3]{(a+b+d)(c+e+f)}.$$

Proposed by Dimitar Trenevski.

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4 Olja writes down n positive integers $a_1, a_2, \ldots, a_n$ smaller than p_n where p_n denotes the n-th prime number. Oleg can choose two (not necessarily different) numbers x and y and replace one of them with their product xy. If there are two equal numbers Oleg wins. Can Oleg guarantee a win?

Proposed by Matko Ljulj.
