European Mathematical Cup 2013

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by Sayan, joybangla, parmenides51, vyfukas

– Junior Division

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1 For $m \in \mathbb{N}$ define $m?$ be the product of first m primes. Determine if there exists positive integers m, n with the following property :

$$m? = n(n+1)(n+2)(n+3)$$

Proposed by Matko Ljulj

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2 Let P be a point inside a triangle ABC. A line through P parallel to AB meets BC and CA at points L and F, respectively. A line through P parallel to BC meets CA and BA at points M and D respectively, and a line through P parallel to CA meets AB and BC at points N and E respectively. Prove

$$[PDBL] \cdot [PECM] \cdot [PFAN] = 8 \cdot [PFM] \cdot [PEL] \cdot [PDN]$$

Proposed by Steve Dinh

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3 We are given a combination lock consisting of 6 rotating discs. Each disc consists of digits $0, 1, 2, \ldots, 9$ in that order (after digit 9 comes 0). Lock is opened by exactly one combination. A move consists of turning one of the discs one digit in any direction and the lock opens instantly if the current combination is correct. Discs are initially put in the position 000000, and we know that this combination is not correct.

a) What is the least number of moves necessary to ensure that we have found the correct combination?
b) What is the least number of moves necessary to ensure that we have found the correct combination, if we
know that none of the combinations $000000, 111111, 222222, \ldots, 999999$ is correct?

Proposed by Ognjen Stipetić and Grgur Valentić

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4 Let a, b, c be positive reals satisfying :

$$\frac{a}{1+b+c} + \frac{b}{1+c+a} + \frac{c}{1+a+b} \geq \frac{ab}{1+a+b} + \frac{bc}{1+b+c} + \frac{ca}{1+c+a}$$

Then prove that :

$$\frac{a^2+b^2+c^2}{ab+bc+ca} + a + b + c + 2 \geq 2(\sqrt{ab} + \sqrt{bc} + \sqrt{ca})$$

Proposed by Dimitar Trenevski

– Senior Division

1 In each field of a table there is a real number. We call such $n \times n$ table *silly* if each entry equals the product of all the numbers in the neighbouring fields.

a) Find all 2×2 silly tables.
b) Find all 3×3 silly tables.

2 Palindrome is a sequence of digits which doesn't change if we reverse the order of its digits. Prove that a sequence $(x_n)_{n=0}^{\infty}$ defined as

$x_n = 2013 + 317n$

contains infinitely many numbers with their decimal expansions being palindromes.

3 We call a sequence of n digits one or zero a code. Subsequence of a code is a palindrome if it is the same after we reverse the order of its digits. A palindrome is called nice if its digits occur consecutively in the code. (Code (1101) contains 10 palindromes, of which 6 are nice.)

a) What is the least number of palindromes in a code?

b) What is the least number of nice palindromes in a code?

4 Given a triangle ABC let D, E, F be orthogonal projections from A, B, C to the opposite sides respectively. Let X, Y, Z denote midpoints of AD, BE, CF respectively. Prove that perpendiculars from D to YZ, from E to XZ and from F to XY are concurrent.