Problems from Year 32 of the USAMTS.

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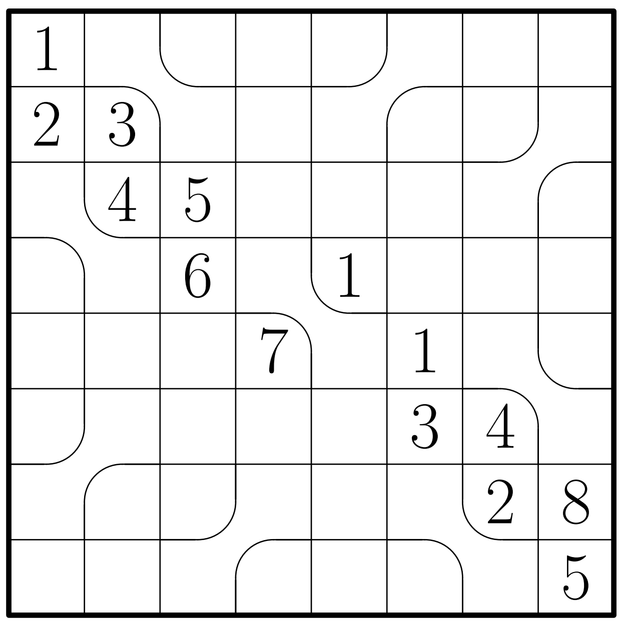
– Round 1

1: Fill in each empty cell of the grid with a digit from 1 to 8 so that every row and every column contains each of these digits exactly once. Some diagonally adjacent cells have been joined together. For these pairs of joined cells, the same number must be written in both.

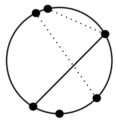


There is a unique solution, but you do not need to prove that your answer is the only one possible. You merely need to find an answer that satisfies the constraints above. (Note: In any other USAMTS problem, you need to provide a full proof. Only in this problem is an answer without justification acceptable.)

2: **2/1/32.** Is it possible to fill in a 2020 x 2020 grid with the integers from 1 to $4,080,400$ so that the sum of each row is 1 greater than the previous row?

3: **3/1/32.** The bisectors of the internal angles of parallelogram $ABCD$ determine a quadrilateral with the same area as $ABCD$. Given that $AB > BC$, compute, with proof, the ratio $\frac{AB}{BC}$.

4: Two beasts, Rosencrans and Gildenstern, play a game. They have a circle with n points ($n \geq 5$) on it. On their turn, each beast (starting with Rosencrans) draws a chord between a pair of points in such a way that any two chords have a shared point. (The chords either intersect or have a common endpoint.) For example, two potential legal moves for the second player are drawn below with dotted lines.



The game ends when a player cannot draw a chord. The last beast to draw a chord wins. For which n does Rosencrans win?

5: **5/1/32.** Find all pairs of rational numbers (a, b) such that $0 < a < b$ and $a^a = b^b$.

– Round 2

1: In the grid below, fill each gray cell with one of the numbers from the provided bank, with each number used once, and fill each white cell with a positive one-digit number. The number in a gray cell must equal the sum of the numbers in all touching white cells, where two cells sharing a vertex are considered touching. All of the terms in each of these sums must be distinct, meaning that two white cells with the same digit may not touch the same gray cell.
Bank: 15, 23, 28, 35, 36, 38, 40, 42, 44

	5				8	
4						
	3					1
					7	
	8			1		4
					2	

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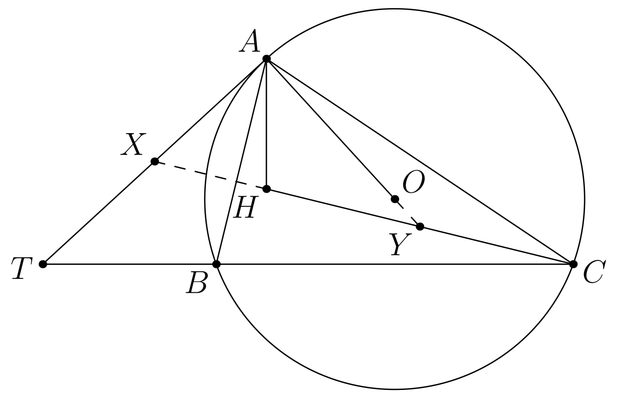
2: Infinitely many math beasts stand in a line, all six feet apart, wearing masks, and with clean hands. Grogg starts at the front of the line, holding n pieces of candy, $n \geq 1$, and everyone else has none. He passes his candy to the beasts behind him, one piece each to the next n beasts in line. Then, Grogg leaves the line. The other beasts repeat this process: the beast in front, who has k pieces of candy, passes one piece each to the next k beasts in line, and then leaves the line. For some values of n, another beast, besides Grogg, temporarily holds all the candy. For which values of n does this occur?

3: Given a nonconstant polynomial with real coefficients $f(x)$, let $S(f)$ denote the sum of its roots. Let p and q be nonconstant polynomials with real coefficients such that $S(p) = 7$, $S(q) = 9$, and $S(p-q) = 11$. Find, with proof, all possible values for $S(p+q)$.

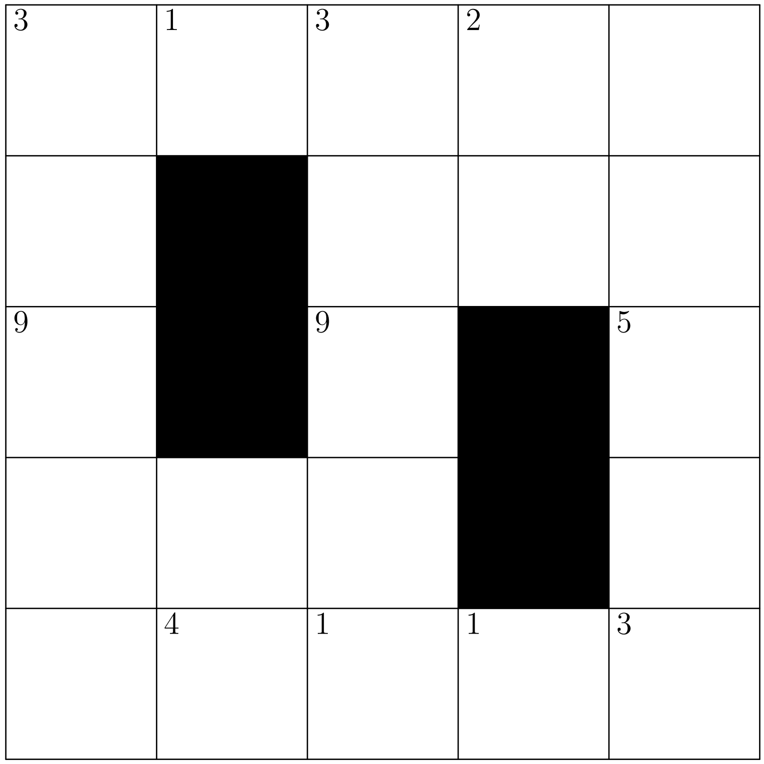
4: Let ABC be a triangle with $AB < AC$. As shown below, T is the point on $\overline{BC}$ such that $\overline{AT}$ is tangent to the circumcircle of $\triangle ABC$. Additionally, H and O are the orthocenter and circumcenter of $\triangle ABC$, respectively. Suppose that $\overline{CH}$ passes through the midpoint of $\overline{AT}$. Prove that $\overline{AO}$ bisects $\overline{CH}$.



5: Let a_1 be any positive integer. For all i, write 5^{2020} times a_i in base 10, replace each digit with its remainder when divided by 2, read off the result in binary, and call that a_{i+1}. Prove that $a_N = a_{N+2^{2020}}$ for all sufficiently large N.

– Round 3

1: Place the 21 two-digit prime numbers in the white squares of the grid on the right so that each two-digit prime is used exactly once. Two white squares sharing a side must contain two numbers with either the same tens digit or ones digit. A given digit in a white square must equal at least one of the two digits of that square's prime number.



There is a unique solution, but you do not need to prove that your answer is the only one possible. You merely need to find an answer that satisfies the constraints above.
(Note: in any other USAMTS problem, you need to provide a full proof. Only in this problem is an answer without justification acceptable.)

2: Find distinct points $A, B, C,$ and D in the plane such that the length of the segment AB is an even integer, and the lengths of the segments $AC, AD, BC, BD,$ and CD are all odd integers. In addition to stating the coordinates of the points and distances between points, please include a brief explanation of how you found the configuration of points and computed the distances.

3: Find, with proof, all positive integers n with the following property: There are only finitely many positive multiples of n which have exactly n positive divisors

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4: In a group of $n > 20$ people, there are some (at least one, and possibly all) pairs of people that know each other. Knowing is symmetric; if Alice knows Blaine, then Blaine also knows Alice. For some values of n and $k,$ this group has a peculiar property: If any 20 people are removed from the group, the number of pairs of people that know each other is
at most $\frac{n-k}{n}$ times that of the original group of people.
(a) If $k = 41,$ for what positive integers n could such a group exist?

(b) If $k = 39,$ for what positive integers n could such a group exist?

5: Let $n \geq 3$ be an integer. Let f be a function from the set of all integers to itself with the following property: If the integers $a_1, a_2, \ldots, a_n$ form an arithmetic progression, then the numbers

$$f(a_1), f(a_2), \ldots, f(a_n)$$

form an arithmetic progression (possibly constant) in some order. Find all values for n such that the only functions f with this property are the functions of the form $f(x) = cx + d$, where c and d are integers.
