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– Individual

I1 A nine-digit number has the form $\overline{6ABCDEFG3}$, where every three consecutive digits sum to 13. Find D.

Proposed by Levi Iszler

I2 Let b and c be real numbers not both equal to 1 such that $1, b, c$ is an arithmetic progression and $1, c, b$ is a geometric progression. What is $100(b-c)$?

Proposed by Noah Kravitz

I3 Suppose that three prime numbers $p, q,$ and r satisfy the equations $pq + qr + rp = 191$ and $p + q = r - 1$. Find $p + q + r$.

Proposed by Andrew Wu

I4 Let $ABCD$ be a square of side length 4. Points E and F are chosen on sides BC and DA, respectively, such that $EF = 5$. Find the sum of the minimum and maximum possible areas of trapezoid $BEDF$.

Proposed by Andrew Wu

I5 For some positive integers $m > n$, the quantities $a = \text{lcm}(m, n)$ and $b = \gcd(m, n)$ satisfy $a = 30b$. If $m - n$ divides a, then what is the value of $\frac{m+n}{b}$?

Proposed by Andrew Wu

I6 Prair has a box with some combination of red and green balls. If she randomly draws two balls out of the box (without replacement), the probability of drawing two balls of the same color is equal to the probability of drawing two balls of different colors! How many possible values between 200 and 1000 are there for the total number of balls in the box?

Proposed by Andrew Wu

I7 Suppose that ABC is a triangle with $AB = 6, BC = 12$, and $\angle B = 90^\circ$. Point D lies on side BC, and point E is constructed on AC such that $\angle ADE = 90^\circ$. Given that $DE = EC = \frac{a\sqrt{b}}{c}$ for positive integers $a, b,$ and c with b squarefree and $\gcd(a, c) = 1$, find $a + b + c$.

Proposed by Andrew Wu

I8 Let $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ be sequences such that $a_ib_i - a_i - b_i = 0$ and $a_{i+1} = \frac{2-a_ib_i}{1-b_i}$ for all $i \geq 1$. If $a_1 = 1 + \frac{1}{\sqrt[4]{2}}$, then what is b_6?

Proposed by Andrew Wu

I9 In how many ways can Irena can write the integers 1 through 7 in a line such that whenever she looks at any three consecutive (in the line) numbers, the largest is not in the rightmost position?

Proposed by Noah Kravitz

I10 Let $f(x)$ be a quadratic polynomial such that $f(f(1)) = f(-f(-1)) = 0$ and $f(1) \neq -f(-1)$. Suppose furthermore that the quadratic $2f(x)$ has coefficients that are nonzero integers. Find $f(0)$.

Proposed by Andrew Wu

I11 Let triangle $\triangle ABC$ have side lengths $AB = 7, BC = 8$, and $CA = 9$, and let M and D be the midpoint of $\overline{BC}$ and the foot of the altitude from A to $\overline{BC}$, respectively. Let E and F lie on $\overline{AB}$ and $\overline{AC}$, respectively, such that $m\angle AEM = m\angle AFM = 90^\circ$. Let P be the intersection of the angle bisectors of $\angle AED$ and $\angle AFD$. If MP can be written as $\frac{a\sqrt{b}}{c}$ for positive integers a, b, and c with b squarefree and $\gcd(a, c) = 1$, then find $a + b + c$.

Proposed by Andrew Wu

I12 Let $p(x)$ be the monic cubic polynomial with roots $\sin^2(1^\circ)$, $\sin^2(3^\circ)$, and $\sin^2(9^\circ)$. Suppose that $p\left(\frac{1}{4}\right) = \frac{\sin(a^\circ)}{n\sin(b^\circ)}$, where $0 < a, b \leq 90$ and a, b, n are positive integers. What is $a + b + n$?

Proposed by Andrew Yuan

– Mathathon Round

– Round 1

p1. Let n be a two-digit positive integer. What is the maximum possible sum of the prime factors of $n^2 - 25$?

p2. Angela has ten numbers $a_1, a_2, a_3, ..., a_{10}$. She wants them to be a permutation of the numbers $\{1, 2, 3, ..., 10\}$ such that for each $1 \leq i \leq 5$, $a_i \leq 2i$, and for each $6 \leq i \leq 10$, $a_i \leq -10$. How many ways can Angela choose a_1 through a_{10}?

p3. Find the number of three-by-three grids such that • the sum of the entries in each row, column, and diagonal passing through the center square is the same, and • the entries in the nine squares are the integers between 1 and 9 inclusive, each integer appearing in exactly one square.

Round 2

p4. Suppose that $P(x)$ is a quadratic polynomial such that the sum and product of its two roots are equal to each other. There is a real number a that $P(1)$ can never be equal to. Find a.

p5. Find the number of ordered pairs (x, y) of positive integers such that $\frac{1}{x}+\frac{1}{y}=\frac{1}{k}$ and k is a factor of 60.

p6. Let ABC be a triangle with $AB = 5$, $AC = 4$, and $BC = 3$. With $B = B_0$ and $C = C_0$, define the infinite sequences of points $\{B_i\}$ and $\{C_i\}$ as follows: for all $i \ge 1$, let B_i be the foot of the perpendicular from C_{i-1} to AB, and let C_i be the foot of the perpendicular from B_i to AC. Find $C_0C_1(AC_0 + AC_1 + AC_2 + AC_3 + ...)$.

Round 3

p7. If $\ell_1, \ell_2, ..., \ell_{10}$ are distinct lines in the plane and $p_1, ..., p_{100}$ are distinct points in the plane, then what is the maximum possible number of ordered pairs (ℓ_i, p_j) such that p_j lies on ℓ_i?

p8. Before Andres goes to school each day, he has to put on a shirt, a jacket, pants, socks, and shoes. He can put these clothes on in any order obeying the following restrictions: socks come before shoes, and the shirt comes before the jacket. How many distinct orders are there for Andres to put his clothes on?

p9. There are ten towns, numbered 1 through 10, and each pair of towns is connected by a road. Define a backwards move to be taking a road from some town a to another town b such that $a > b$, and define a forwards move to be taking a road from some town a to another town b such that $a < b$. How many distinct paths can Ali take from town 1 to town 10 under the conditions that • she takes exactly one backwards move and the rest of her moves are forward moves, and • the only time she visits town 10 is at the very end?
One possible path is $1 \to 3 \to 8 \to 6 \to 7 \to 8 \to 10$.

Round 4

p10. How many prime numbers p less than 100 have the properties that $p^5 - 1$ is divisible by 6 and $p^6 - 1$ is divisible by 5?

p11. Call a four-digit integer $\overline{d_1d_2d_3d_4}$ *primed* if
1) d_1, d_2, d_3, and d_4 are all prime numbers, and
2) the two-digit numbers $\overline{d_1d_2}$ and $\overline{d_3d_4}$ are both prime numbers.
Find the sum of all primed integers.

p12. Suppose that ABC is an isosceles triangle with $AB = AC$, and suppose that D and E lie on $\overline{AB}$ and $\overline{AC}$, respectively, with $\overline{DE} \parallel \overline{BC}$. Let r be the length of the inradius of triangle ADE. Suppose that it is possible to construct two circles of radius r, each tangent to one another and internally tangent to three sides of the trapezoid $BDEC$. If $\frac{BC}{r} = a + \sqrt{b}$ forpositive integers a and b with b squarefree, then find $a + b$.

PS. You should use hide for answers. Rounds 5-7 have been posted here (`https://artofproblemsolving.com/community/c4h2800986p24675177`). Collected here (`https://artofproblemsolving.com/community/c5h2760506p24143309`).

– Round 5

p13. A palindrome is a number that reads the same forward as backwards; for example, 121 and 36463 are palindromes. Suppose that N is the maximal possible difference between two consecutive three-digit palindromes. Find the number of pairs of consecutive palindromes (A, B) satisfying $A < B$ and $B - A = N$.

p14. Suppose that x, y, and z are complex numbers satisfying $x + \frac{1}{yz} = 5$, $y + \frac{1}{zx} = 8$, and $z + \frac{1}{xy} = 6$. Find the sum of all possible values of xyz.

p15. Let Ω be a circle with radius $25\sqrt{2}$ centered at O, and let C and J be points on Ω such that the circle with diameter $\overline{CJ}$ passes through O. Let Q be a point on the circle with diameter $\overline{CJ}$ satisfying $OQ = 5\sqrt{2}$. If the area of the region bounded by $\overline{CQ}$, $\overline{QJ}$, and minor arc JC on Ω can be expressed as $\frac{a\pi - b}{c}$ for integers a, b, and c with $gcd\ (a, c) = 1$, then find $a + b + c$.

Round 6

p16. Veronica writes N integers between 2 and 2020 (inclusive) on a blackboard, and she notices that no number on the board is an integer power of another number on the board. What is the largest possible value of N?

p17. Let ABC be a triangle with $AB = 12$, $AC = 16$, and $BC = 20$. Let D be a point on AC, and suppose that I and J are the incenters of triangles ABD and CBD, respectively. Suppose that

$DI = DJ$. Find IJ^2.

p18. For each positive integer a, let $P_a = \{2a, 3a, 5a, 7a, ...\}$ be the set of all prime multiples of a. Let $f(m, n) = 1$ if P_m and P_n have elements in common, and let $f(m, n) = 0$ if P_m and P_n have no elements in common. Compute

$$\sum_{1\le i<j\le 50} f(i, j)$$

(i.e. compute $f(1, 2) + f(1, 3)+, , , +f(1, 50) + f(2, 3) + f(2, 4)+, , , +f(49, 50)$.)

Round 7

p19. How many ways are there to put the six letters in "$MMATHS$" in a two-by-three grid such that the two "M"s do not occupy adjacent squares and such that the letter "A" is not directly above the letter "T" in the grid? (Squares are said to be adjacent if they share a side.)

p20. Luke is shooting basketballs into a hoop. He makes any given shot with fixed probability p with $p < 1$, and he shoots n shots in total with $n \ge 2$. Miraculously, in n shots, the probability that Luke makes exactly two shots in is twice the probability that Luke makes exactly one shot in! If p can be expressed as $\frac{k}{100}$ for some integer k (not necessarily in lowest terms), find the sum of all possible values for k.

p21. Let $ABCD$ be a rectangle with $AB = 24$ and $BC = 72$. Call a point P *goofy* if it satisfies the following conditions: • P lies within $ABCD$, • for some points F and G lying on sides BC and DA such that the circles with diameter BF and DG are tangent to one another, P lies on their common internal tangent.
Find the smallest possible area of a polygon that contains every single goofy point inside it.

PS. You should use hide for answers. Rounds 1-4 have been posted here (`https://artofproblemsolving.com/community/c4h2800971p24674988`). Collected here (`https://artofproblemsolving.com/community/c5h2760506p24143309`).

– Tiebreaker

1 A positive integer n is called an untouchable number if there is no positive integer m for which the sum of the factors of m (including m itself) is $n + m$. Find the sum of all of the untouchable numbers between 1 and 10 (inclusive)

2 Suppose that points A and B lie on circle Ω, and suppose that points C and D are the trisection points of major arc AB, with C closer to B than A. Let E be the intersection of line AB with the

line tangent to Ω at C. Suppose that $DC = 8$ and $DB = 11$. If $DE = a\sqrt{b}$ for integers a and b with b squarefree, find $a + b$.

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3 Let a, b be two real numbers such that

$$\sqrt[3]{a} - \sqrt[3]{b} = 10, , \quad ab = \left(\frac{8-a-b}{6}\right)^3$$

Find $a - b$.

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4 Define the function $f(n)$ for positive integers n as follows: if n is prime, then $f(n) = 1$; and $f(ab) = a \cdot f(b) + f(a) \cdot b$ for all positive integers a and b. How many positive integers n less than 5^{50} have the property that $f(n) = n$?

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5 Let x, y be positive reals such that $x \neq y$. Find the minimum possible value of $(x+y)^2 + \frac{54}{xy(x-y)^2}$.

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6 Consider the function $f(n) = n^2 + n + 1$. For each n, let d_n be the smallest positive integer with $gcd(n, dn) = 1$ and $f(n)|f(d_n)$. Find $d_6 + d_7 + d_8 + d_9 + d_{10}$.

– Mixer Round

– **p1.** There are five boys and five girls in a class. Suppose that they pair off in the following manner: two boys pair, two girls pair, and the rest of the class divides into boy-girl pairs. How many ways are there for this to occur?

p2. Annie and Britta are playing a game. Annie picks a triple of single-digit positive integers and gives Britta the following hints: 1) Her numbers are all distinct; 2) The sum of her numbers is greater than or equal to 20; 3) The product of her numbers is divisible by 32. Britta computes the sum of all single-digit positive integers that Annie didn't choose. What number did Britta compute?

p3. 8 ants, uncreatively denoted by Ant_1, Ant_2,..., Ant_8, must each stand at a vertex of a regular octagon such that each ant stands at exactly one vertex, and the line segments joining Ant_i and Ant_{i+1} for $i = 1$,2, 3, ..., 7 do not intersect except at the vertices of the octagon. How many ways can they do this? (Rotations and reflections are considered distinct.)

p4. Claire and Cat are at a party with 10 other people. Every 10 minutes, two random people get up and leave, and this continues for an hour (until everybody has left). If the probability that Claire leaves before Cat can be expressed as $\frac{a}{b}$ in simplest terms, find $a + b$.

p5. Given that integers a and b with $a + b \neq 0$ satisfy the equation $a + b - 2 = \frac{a+1}{b} + \frac{b+1}{a}$, find the sum of all possible values of a.

p6. Suppose that Ω is a circle with center O, and let A, B, and C be points on Ω satisfying $AB = AC$. Suppose that there exists a circle ω centered at O tangent to $\overline{AB}$, $\overline{AC}$, and the arc BC of the circle centered at A passing through B and C. If the ratio of the area of ω to the area of Ω can be expressed as $\frac{a}{b}$ for integers a, b satisfying $gcd(a, b) = 1$, then find $a + b$.

p7. Jennifer rolls three fair six-sided dice, with integer labels from 1 to 6. Let N be the product of the numbers she rolls; i.e., if she rolls 1, 4, and 5, then $N = 20$. If the probability that N has exactly 8 factors can be expressed as $\frac{a}{b}$ in simplest terms, find $a + b$.

p8. How many ways are there to arrange four rooks on an four-by-four chessboard such that exactly two pairs of rooks attack each other? (Rooks are said to be attacking one another if they lie in the same row or column, without any rooks in between them.)

p9. Let ABC be an equilateral triangle, and consider points D, E on line BC such that $BD = CE = \frac{BC}{3}$ and $DE = BC$. The value of $\sin \angle DAE$ can be expressed as $a\sqrt{b}c$, where a, b, c are positive integers, b is squarefree, and $gcd(a, c) = 1$. Find $a + b + c$.

p10. Given an integer n, $f(n)$ returns the number of distinct positive integers a such that $\frac{na}{n+a}$ is an integer. If $A = f(3) + f(6) + f(12) + ... + f(3 \cdot 2^k) + ... + f(3 \cdot 2^{673})$, find the remainder when A is divided by 2020.

p11. Let a, b be two real numbers such that

$$\sqrt[3]{a} - \sqrt[3]{b} = 10, \qquad ab = \left(\frac{8 - a - b}{6}\right)^3.$$

Find $a - b$.

p12. Consider the function $f(n) = n^2 + n + 1$. For each n, let d_n be the smallest positive integer with $gcd(n, dn) = 1$ and $f(n)j|f(d_n)$. Find $d_6 + d_7 + d_8 + d_9 + d_{10}$.

p13. Suppose that ABC is a triangle with $AB = 13$, $BC = 14$, and $CA = 15$; let M and N be the midpoints of AB and AC, and suppose that D is the foot of the A-altitude. Point X is chosen on the circumcircle of triangle DMN such that if Y is the reflection of X in MN, the length AY is maximal among all possible choices of X. If $AX = \frac{\sqrt{a}}{b}$ for integers a and b with a squarefree, find $a + b$.

p14. Let x, y be positive reals such that $x \neq y$. Find the minimum possible value of $(x+y)^2 + \frac{54}{xy(x-y)^2}$.

p15. Define the function $f(n)$ for positive integers n as follows: if n is prime, then $f(n) = 1$, and

$$f(ab) = a \cdot f(b) + f(a) \cdot b$$

for all positive integers a and b. How many positive integers n less than 5^{50} have the property that $f(n) = n$?

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