Problems from the 2020-2021 Fall SDPC. Middle School division does 1,2,3,5,6,7, High School division does 2,3,4,6,7,8.

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by mira74

– Session 1

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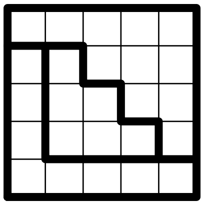
1 In the following grid below, each row and column contains the numbers $1, 2, 3, 4, 5$ exactly once. Furthermore, each of the three sections have the same sum. Find, with proof, all possible ways to fill the grid in.



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2 Let $k > 1$ be a positive integer. On a k $\times$ k square grid, Tom and Jerry are on opposite corners, with Tom at the top right corner. Both can move to an adjacent square every move, where two squares are adjacent if they share a side. Tom and Jerry alternate moves, with Jerry going first. Tom *catches* Jerry if they are on the same square. We aim to answer to the following question: What is the smallest number of moves that Tom needs to guarantee catching Jerry?

(a) Without proof, find the answer in the cases of $k = 2, 3, 4$, and (correctly) guess what the answer is in terms of k. We'll refer to this answer as $A(k)$.

(b) Find a strategy that Jerry can use to guarantee that Tom takes at least $A(k)$ moves to catch Jerry.

Now, you will find a strategy for Tom to catch Jerry in at most $A(k)$ moves, no matter what Jerry does.

(c) Find, with proof, a working strategy for $k = 5$.

(d) Find, with proof, a working strategy for all $k \geq 2$.

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3 For some fixed positive integer $n > 2$, suppose $x_1, x_2, x_3, \ldots$ is a nonconstant sequence of real numbers such that $x_i = x_j$ if $i \equiv j \pmod{n}$. Let $f(i) = x_i + x_i x_{i+1} + \cdots + x_i x_{i+1} \ldots x_{i+n-1}$. Given that

$$f(1) = f(2) = f(3) = \cdots$$

find all possible values of the product $x_1 x_2 \ldots x_n$.

4 Let ABC be an acute scalene triangle, let D be a point on the A-altitude, and let the circle with diameter AD meet AC, AB, and the circumcircle of ABC at E, F, G, respectively. Let O be the circumcenter of ABC, let AO meet EF at T, and suppose the circumcircles of ABC and GTO meet at $X \neq G$. Then, prove that AX, DG, and EF concur.

– Session 2

5 Let ABC be a triangle with area 1. Let D be a point on segment BC. Let points E and F on AC and AB, respectively, satisfy $DE||AB$ and $DF||AC$. Compute, with proof, the area of the quadrilateral with vertices at E, F, the midpoint of BD, and the midpoint of CD.

6 For a positive integer n, let $f(n)$ be the greatest common divisor of all numbers obtained by permuting the digits of n, including the permutations that have leading zeroes. For example, $f(1110) = \gcd(1110, 1101, 1011, 0111) = 3$. Among all positive integers n with $f(n) \neq n$, what is the largest possible value of $f(n)$?

7 Alice is wandering in the country of Wanderland. Wanderland consists of a finite number of cities, some of which are connected by two-way trains, such that Wanderland is connected: given any two cities, there is always a way to get from one to the other through a series of train rides.

Alice starts at Riverbank City and wants to end up at Conscious City. Every day, she picks a train going out of the city she is in uniformly at random among all of the trains, and then boards that train to the city it leads to. Show that the expected number of days it takes for her to reach Conscious City is finite.

8 Let $\mathbb{R}$ denote the set of all real numbers. Find all functions f from $\mathbb{R}$ to $\mathbb{R}$ such that

$$x^2 f(x^2 + y^2) + y^4 = (xf(x + y) + y^2)(xf(x - y) + y^2)$$

for all $x, y \in \mathbb{R}$.

