Hong Kong Team Selection Tests for IMO 2021

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– **Test 1** October 17, 2020, 3 hours

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1 Find all real triples (a,b,c) satisfying
$$(2^{2a}+1)(2^{2b}+2)(2^{2c}+8)=2^{a+b+c+5}.$$

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2 In ΔABC, $AC=kAB$, with $k>1$. The internal angle bisector of $\angle BAC$ meets BC at D. The circle with AC as diameter cuts the extension of AD at E. Express $\dfrac{AD}{AE}$ in terms of k.

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3 On the table there are 20 coins of weights $1,2,3,\dots,15,37,38,39,40$ and 41 grams. They all look alike but their colours are all distinct. Now Miss Adams knows the weight and colour of each coin, but Mr. Bean knows only the weights of the coins. There is also a balance on the table, and each comparison of weights of two groups of coins is called an operation. Miss Adams wants to tell Mr. Bean which coin is the 1 gram coin by performing some operations. What is the minimum number of operations she needs to perform?

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4 Let n be a positive integer. Is it possible to express n^2+3n+3 into the form ab with a and b being positive integers, and such that the difference between a and b is smaller than $2\sqrt{n+1}$?

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5 Let $ABCD$ be an isosceles trapezoid with base BC and AD. Suppose $\angle BDC=10^\circ$ and $\angle BDA=70^\circ$. Show that $AD^2=BC(AD+AB)$.

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6 There is an $n\times n$ chessboard where $n\geq 4$ is a positive even number. The cells of the chessboard are coloured black and white such that adjacent cells sharing a common side have different colours. Let A and B be two interior cells (which means cells not lying on an edge of the chessboard) of distinct colours. Prove that a chess piece can move from A to B by moving across adjacent cells such that every cell of the chessboard is passed through exactly once.

– **Test 2** October 31, 2020, 3 hours

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1 Let S be a set of 2020 distinct points in the plane. Let
$$M=\{P:P\text{ is the midpoint of }XY\text{ for some distinct points }X,Y\text{ in }S\}.$$

Find the least possible value of the number of points in M.

2 Let $f(x)$ be a polynomial with rational coefficients, and let α be a real number. If

$$\alpha^3 - 2019\alpha = (f(\alpha))^3 - 2019f(\alpha) = 2021,$$

prove that $(f^n(\alpha))^3 - 2019f^n(\alpha) = 2021$ for any positive integer n.

(Here, we define $f^n(x) = \underbrace{f(f(f \cdots f}_{n \text{ times}}(x) \cdots))$.)

3 Let $\triangle ABC$ be an acute triangle with circumcircle Γ, and let P be the midpoint of the minor arc BC of Γ. Let AP and BC meet at D, and let M be the midpoint of AB. Also, let E be the point such that $AE \perp AB$ and $BE \perp MP$. Prove that $AE = DE$.

4 Does there exist a nonzero polynomial $P(x)$ with integer coefficients satisfying both of the following conditions?

-$P(x)$ has no rational root;
-For every positive integer n, there exists an integer m such that n divides $P(m)$.

Test 3 May 1, 2021, 4.5 hours

Problem 1 (Released after IMO 2021)

Problem 2 (Released after IMO 2021)

Problem 3 (Released after IMO 2021)

Test 4 May 2, 2021, 4.5 hours

Problem 4 (Released after IMO 2021)

Problem 5 (Released after IMO 2021)

Problem 6 (Released after IMO 2021)