Balkan MO 2020

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1	Let ABC be an acute triangle with $AB = AC$, let D be the midpoint of the side AC, and let γ be the circumcircle of the triangle ABD. The tangent of γ at A crosses the line BC at E. Let O be the circumcenter of the triangle ABE. Prove that midpoint of the segment AO lies on γ. *Proposed by Sam Bealing, United Kingdom*
2	Denote $\mathbb{Z}_{>0} = \{1, 2, 3, ...\}$ the set of all positive integers. Determine all functions $f : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ such that, for each positive integer n, $i) \sum_{k=1}^{n} f(k)$ is a perfect square, and $ii) f(n)$ divides n^3. *Proposed by Dorlir Ahmeti, Albania*
3	Let k be a positive integer. Determine the least positive integer n, with $n \geq k + 1$, for which the game below can be played indefinitely: Consider n boxes, labelled $b_1, b_2, ..., b_n$. For each index i, box b_i contains exactly i coins. At each step, the following three substeps are performed in order: **(1)** Choose $k + 1$ boxes; **(2)** Of these $k + 1$ boxes, choose k and remove at least half of the coins from each, and add to the remaining box, if labelled b_i, a number of i coins. **(3)** If one of the boxes is left empty, the game ends; otherwise, go to the next step. *Proposed by Demetres Christofides, Cyprus*
4	Let $a_1 = 2$ and, for every positive integer n, let a_{n+1} be the smallest integer strictly greater than a_n that has more positive divisors than a_n. Prove that $2a_{n+1} = 3a_n$ only for finitely many indicies n. *Proposed by Ilija Jovčevski, North Macedonia*