

**2020 United States Ersatz Math Olympiad, 24-25 October 2020**
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**Day 1** October 24, 2020

- 1 Which positive integers can be written in the form

$$\frac{\text{lcm}(x, y) + \text{lcm}(y, z)}{\text{lcm}(x, z)}$$

 for positive integers  $x, y, z$ ?

- 2 Calvin and Hobbes play a game. First, Hobbes picks a family  $F$  of subsets of  $\{1, 2, \dots, 2020\}$ , known to both players. Then, Calvin and Hobbes take turns choosing a number from  $\{1, 2, \dots, 2020\}$  which is not already chosen, with Calvin going first, until all numbers are taken (i.e., each player has 1010 numbers). Calvin wins if he has chosen all the elements of some member of  $F$ , otherwise Hobbes wins. What is the largest possible size of a family  $F$  that Hobbes could pick while still having a winning strategy?

- 3 Let  $ABC$  be an acute triangle with circumcenter  $O$  and orthocenter  $H$ . Let  $\Gamma$  denote the circumcircle of triangle  $ABC$ , and  $N$  the midpoint of  $OH$ . The tangents to  $\Gamma$  at  $B$  and  $C$ , and the line through  $H$  perpendicular to line  $AN$ , determine a triangle whose circumcircle we denote by  $\omega_A$ . Define  $\omega_B$  and  $\omega_C$  similarly. Prove that the common chords of  $\omega_A, \omega_B$  and  $\omega_C$  are concurrent on line  $OH$ .

Proposed by Anant Mudgal

**Day 2** October 25, 2020

- 4 A function  $f$  from the set of positive real numbers to itself satisfies

$$f(x + f(y) + xy) = xf(y) + f(x + y)$$

 for all positive real numbers  $x$  and  $y$ . Prove that  $f(x) = x$  for all positive real numbers  $x$ .

- 5 The sides of a convex 200-gon  $A_1A_2 \dots A_{200}$  are colored red and blue in an alternating fashion. Suppose the extensions of the red sides determine a regular 100-gon, as do the extensions of the blue sides.

 Prove that the 50 diagonals  $\overline{A_1A_{101}}, \overline{A_3A_{103}}, \dots, \overline{A_{99}A_{199}}$  are concurrent.

 Proposed by: **Ankan Bhattacharya**

- 6 Prove that for every odd integer  $n > 1$ , there exist integers  $a, b > 0$  such that, if we let  $Q(x) = (x + a)^2 + b$ , then the following conditions hold: • we have  $\gcd(a, n) = \gcd(b, n) = 1$ ; • the number  $Q(0)$  is divisible by  $n$ ; and • the numbers  $Q(1), Q(2), Q(3), \dots$  each have a prime factor not dividing  $n$ .
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