Kazakhstan National Olympiad 2017

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– grade 11

– day 1

1 The non-isosceles triangle ABC is inscribed in the circle ω. The tangent line to this circle at the point C intersects the line AB at the point D. Let the bisector of the angle CDB intersect the segments AC and BC at the points K and L, respectively. The point M is on the side AB such that $\frac{AK}{BL}=\frac{AM}{BM}$. Let the perpendiculars from the point M to the straight lines KL and DC intersect the lines AC and DC at the points P and Q respectively. Prove that $2\angle CQP=\angle ACB$

2 For positive reals $x,y,z\ge \frac{1}{2}$ with $x^2+y^2+z^2=1$, prove this inequality holds

$$(\frac{1}{x}+\frac{1}{y}-\frac{1}{z})(\frac{1}{x}-\frac{1}{y}+\frac{1}{z})\ge 2$$

3 $\{a_n\}$ is an infinite, strictly increasing sequence of positive integers and $a_{a_n}\le a_n+a_{n+3}$ for all $n\ge 1$. Prove that, there are infinitely many triples (k,l,m) of positive integers such that $k<l<m$ and $a_k+a_m=2a_l$

– day 2

4 The acute triangle ABC $(AC>BC)$ is inscribed in a circle with the center at the point O, and CD is the diameter of this circle. The point K is on the continuation of the ray DA beyond the point A. And the point L is on the segment BD $(DL>LB)$ so that $\angle OKD=\angle BAC$, $\angle OLD=\angle ABC$. Prove that the line KL passes through the midpoint of the segment AB.

5 Consider all possible sets of natural numbers $(x_1,x_2,...,x_{100})$ such that $1\le x_i\le 2017$ for every $i=1,2,...,100$. We say that the set $(y_1,y_2,...,y_{100})$ is greater than the set $(z_1,z_2,...,z_{100})$ if $y_i>z_i$ for every $i=1,2,...,100$. What is the largest number of sets that can be written on the board, so that any set is not more than the other set?

6 Show that there exist infinitely many composite positive integers n such that n divides $2^{\frac{n-1}{2}}+1$