Kazakhstan National Olympiad 2000

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–	grade 11
–	day 1
1	Two guys are playing the game "Sea Battle-2000". On the board 1×200, they take turns placing the letter "S" or "O" on the empty squares of the board. The winner is the one who gets the word "SOS" first. Prove that the second player wins when played correctly.
2	Given a circle centered at O and two points A and B lying on it. A and B do not form a diameter. The point C is chosen on the circle so that the line AC divides the segment OB in half. Let lines AB and OC intersect at D, and let lines BC and AO intersect at F. Prove that $AF = CD$.
3	In a country with n ($n\geq 3$) airports, the government only licenses air travel to those airlines whose airline system meets the following conditions: a) Each airline must connect any two airports with one and only one one-way airline; b) For each airline there is an airport from which the passenger could fly off and fly back, using the services of only this airline. What is the maximum number of airlines with different airline systems?
4	Find all triples of natural numbers (x, y, z) that satisfy the condition $(x + 1)^{y + 1} + 1 = (x + 2)^{z + 1}$.
–	day 2
5	Let the number p be a prime divisor of the number $2^{2^k} + 1$. Prove that $p-1$ is divisible by $2^{k + 1}$.
6	For positive numbers a, b and c satisfying the equality $a+b+c=1$, prove the inequality $$\frac{a^7+b^7}{a^5+b^5}+\frac{b^7+c^7}{b^5+c^5}+\frac{c^7+a^7}{c^5+a^5}\geq\frac{1}{3}.$$
7	Is there any function $f:\mathbb{R}\rightarrow\mathbb{R}$ satisfying following conditions: $1) f(0)=1$ $2) f(x+f(y))=f(x+y)+1$, for all $x,y\rightarrow\mathbb{R}$ $3)$ there exist rational, but not integer x_0, such $f(x_0)$ is integer
8	Given a triangle ABC and a point M inside it. Prove that $$\min\{MA, MB, MC\} + MA + MB + MC < AB + BC + AC.$$