Kazakhstan National Olympiad 2001

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–	grade 11
–	day 1
1	Prove that there are infinitely many natural numbers n such that $2^n + 3^n$ is divisible by n.
2	In the acute triangle ABC, L, H and M are the intersection points of bisectors, altitudes and medians, respectively, and O is the center of the circumscribed circle. Denote by X, Y and Z the intersection points of AL, BL and CL with a circle, respectively. Let N be a point on the line OL such that the lines MN and HL are parallel. Prove that N is the intersection point of the medians of XYZ.
3	For positive numbers $x_1, x_2, \ldots, x_n$ $(n \geq 1)$ the following equality holds $$\frac{1}{1+x_1} + \frac{1}{1+x_2} + \ldots + \frac{1}{1+x_n} = 1.$$ Prove that $x_1 \cdot x_2 \cdot \ldots \cdot x_n \geq (n-1)^n$.
4	Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying the equality $f(x^2 - y^2) = (x - y)(f(x) + f(y))$ for any $x, y \in \mathbb{R}$.
–	day 2
5	Find all possible pairs of real numbers (x, y) that satisfy the equalities $y^2 - [x]^2 = 2001$ and $x^2 + [y]^2 = 2001$.
6	Each interior point of an equilateral triangle with sides equal to 1 lies in one of six circles of the same radius r. Prove that $r \geq \frac{\sqrt{3}}{10}$.
7	Two circles w_1 and w_2 intersect at two points P and Q. The common tangent to w_1 and w_2, which is closer to the point P than to Q, touches these circles at A and B, respectively. The tangent to w_1 at the point P intersects w_2 at the point E (different from P), and the tangent to w_2 at the point P intersects w_1 at F (different from P). Let H and K be points on the rays AF and BE, respectively, such that $AH = AP$ and $BK = BP$. Prove that the points A, H, Q, K and B lie on the same circle.

8 There are $n \geq 4$ points on the plane, the distance between any two of which is an integer. Prove that there are at least $\frac{1}{6}$ distances, each of which is divisible by 3.
