Kazakhstan National Olympiad 2002

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–	grade 11
–	day 1
1	Let O be the center of the inscribed circle of the triangle ABC, tangent to the side of BC. Let M be the midpoint of AC, and P be the intersection point of MO and BC. Prove that $AB = BP$ if $\angle BAC = 2\angle ACB$.
2	Let $x_1, x_2, \ldots, x_n$ be arbitrary real numbers. Prove the inequality $$\frac{x_1}{1+x_1^2} + \frac{x_2}{1+x_1^2+x_2^2} + \cdots + \frac{x_n}{1+x_1^2+\cdots+x_n^2} < \sqrt{n}.$$
3	Let $A = (a_1, a_2, \ldots, a_{2001})$ be a sequence of positive integers. Let m be the number of 3-element subsequences (a_i, a_j, a_k) with $1 \le i < j < k \le 2001$, such that $a_j = a_i + 1$ and $a_k = a_j + 1$. Considering all such sequences A, find the greatest value of m.
4	Prove that there is a set A consisting of 2002 different natural numbers satisfying the condition: for each $a \in A$, the product of all numbers from A, except a, when divided by a gives the remainder 1.
–	day 2
5	On the plane is given the acute triangle ABC. Let A_1 and B_1 be the feet of the altitudes of A and B drawn from those vertices, respectively. Tangents at points A_1 and B_1 drawn to the circumscribed circle of the triangle CA_1B_1 intersect at M. Prove that the circles circumscribed around the triangles AMB_1, BMA_1 and CA_1B_1 have a common point.
6	Find all polynomials $P(x)$ with real coefficients that satisfy the identity $P(x^2) = P(x)P(x+1)$.
7	Prove that for any integers $n > m > 0$ the number $2^n - 1$ has a prime divisor not dividing $2^m - 1$.
8	N grasshoppers are lined up in a row. At any time, one grasshopper is allowed to jump over exactly two grasshoppers standing to the right or left of him. At what n can grasshoppers rearrange themselves in reverse order?