Kazakhstan National Olympiad 2003

www.artofproblemsolving.com/community/c1460040

by parmenides51, Erken, izat, orl, mathlink

– grade 11

– day 1

1 Find all natural numbers n,such that there exist $x_1, x_2, \ldots, x_{n+1} \in \mathbb{N}$,such that $\frac{1}{x_1^2} + \frac{1}{x_2^2} + \cdots + \frac{1}{x_n^2} = \frac{n+1}{x_{n+1}^2}$.

2 For positive real numbers x, y, z, prove the inequality:

$$\frac{x^3}{x+y} + \frac{y^3}{y+z} + \frac{z^3}{z+x} \geq \frac{xy+yz+zx}{2}.$$

3 Two square sheets have areas equal to 2003. Each of the sheets is arbitrarily divided into 2003 nonoverlapping polygons, besides, each of the polygons has an unitary area. Afterward, one overlays two sheets, and it is asked to prove that the obtained double layer can be punctured 2003 times, so that each of the 4006 polygons gets punctured precisely once.

4 Let the inscribed circle ω of triangle ABC touch the side BC at the point A'. Let AA' intersect ω at $P \neq A$. Let CP and BP intersect ω, respectively, at points N and M other than P. Prove that AA', BN and CM intersect at one point.

– day 2

5 Prove that for all primes $p > 3$, $\binom{2p}{p} - 2$ is divisible by p^3

6 Let the point B lie on the circle S_1 and let the point A, other than the point B, lie on the tangent to the circle S_1 passing through the point B. Let a point C be chosen outside the circle S_1, so that the segment AC intersects S_1 at two different points. Let the circle S_2 touch the line AC at the point C and the circle S_1 at the point D, on the opposite side from the point B with respect to the line AC. Prove that the center of the circumcircle of triangle BCD lies on the circumcircle of triangle ABC.

7 For n an odd positive integer, the unit squares of an $n \times n$ chessboard are coloured alternately black and white, with the four corners coloured black. A it tromino is an L-shape formed by three connected unit squares. For which values of n is it possible to cover all the black squares with non-overlapping trominos? When it is possible, what is the minimum number of trominos needed?

---

8 Determine all functions $f:\mathbb{R}\to\mathbb{R}$ with the property

$$f(f(x)+y)=2x+f(f(y)-x), \quad \forall x,y\in\mathbb{R}.$$
