Kazakhstan National Olympiad 2004

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–	grade 11
–	day 1
1	For reals $1 \leq a \leq b \leq c \leq d \leq e \leq f$ prove inequality $(af + be + cd)(af + bd + ce) \leq (a + b^2 + c^3)(d + e^2 + f^3)$.
2	A *zigzag* is a polyline on a plane formed from two parallel rays and a segment connecting the origins of these rays. What is the maximum number of parts a plane can be split into using n zigzags?
3	Does there exist a sequence $\{a_n\}$ of positive integers satisfying the following conditions: $a)$ every natural number occurs in this sequence and exactly once; $b)$ $a_1 + a_2 + ... + a_n$ is divisible by n^n for each $n = 1, 2, 3, ...$?
4	In some village there are 1000 inhabitants. Every day, each of them shares the news they learned yesterday with all their friends. It is known that any news becomes known to all residents of the village. Prove that it is possible to select 90 residents so that if you tell all of them at the same time some news, then in 10 days it will become known to all residents of the village.
–	day 2
5	Let $P(x)$ be a polynomial with real coefficients such that $P(x) > 0$ for all $x \geq 0$. Prove that there is a positive integer n such that $(1+x)^n P(x)$ polynomial with nonnegative coefficients.
6	The sequence of integers $a_1, a_2, \dots$ is defined as follows: $a_1 = 1$ and $n > 1$, a_{n+1} is the smallest integer greater than a_n and such, that $a_i + a_j \neq 3a_k$ for any i, j and k from $\{1, 2, \dots, n+1\}$ are not necessarily different. Define a_{2004}.
7	Prove that for any $a > 0, b > 0, c > 0$ we have $8a^2b^2c^2 \geq (a^2 + ab + ac - bc)(b^2 + ba + bc - ac)(c^2 + ca + cb - ab).$
8	Let $ABCD$ be a convex quadrilateral. The perpendicular bisectors of its sides AB and CD meet at Y. Denote by X a point inside the quadrilateral $ABCD$ such that $\measuredangle ADX = \measuredangle BCX < 90^\circ$

and $\measuredangle DAX = \measuredangle CBX < 90^\circ$. Show that $\measuredangle AYB = 2 \cdot \measuredangle ADX$.