**Kazakhstan National Olympiad 1999**

www.artofproblemsolving.com/community/c1460238

by parmenides51

| | |
|---|---|
| – | grade 11 |
| – | day 1 |
| **1** | Prove that for any real numbers $a_1, a_2, \ldots, a_{100}$ there exists a real number $b$ such that all numbers $a_i + b$ $(1 \le i \le 100)$ are irrational. |
| **2** | Prove that for any odd $n$ there exists a unique polynomial $P(x)$ $n$ -th degree satisfying the equation $P\left(x - \frac{1}{x}\right) = x^n - \frac{1}{x^n}$. Is this true for any natural number $n$? |
| **3** | The circle inscribed in the triangle $ABC$ , with center $O$, touches the sides $AB$ and $BC$ at the points $C_1$ and $A_1$, respectively. The lines $CO$ and $AO$ intersect the line $C_1A_1$ at the points $K$ and $L$. $M$ is the midpoint of $AC$ and $\angle ABC = 60^\circ$. Prove that $KLM$ is a regular triangle. |
| **4** | Seven dwarfs live in one house and each has its own hat. One morning one day, two dwarfs inadvertently exchanged hats. At any time, any three gnomes can sit down at the round table and exchange hats clockwise. Is it possible that by evening all the gnomes will be with their hats. |
| – | day 2 |
| **5** | For real numbers $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ , the inequalities hold $x_1 \ge x_2 \ge \ldots \ge x_n > 0$ and $$y_1 \ge x_1,\ y_1y_2 \ge x_1x_2,\ \ \ldots,\ y_1y_2\ldots y_n \ge x_1x_2\ldots x_n.$$ Prove that $ny_1 + (n-1)y_2 + \cdots + y_n \ge x_1 + 2x_2 + \cdots + nx_n$. |
| **6** | In a sequence of natural numbers $a_1$, $a_2$, $\ldots$, $a_{1999}$, $a_n - a_{n-1} - a_{n-2}$ is divisible by $100 (3 \le n \le 1999)$. It is known that $a_1 = 19$ and $a_2 = 99$. Find the remainder of $a_1^2 + a_2^2 + \cdots + a_{1999}^2$ by $8$. |
| **7** | On a sphere with radius $1$, a point $P$ is given. Three mutually perpendicular the rays emanating from the point $P$ intersect the sphere at the points $A$, $B$ and $C$. Prove that all such possible $ABC$ planes pass through fixed point, and find the maximum possible area of the triangle $ABC$ |
| **8** | Let $a_1, a_2, \ldots, a_n$ be permutation of numbers $1, 2, \ldots, n$, where $n \ge 2$.<br>Find the maximum value of the sum $$S(n) = \lvert a_1 - a_2\rvert + \lvert a_2 - a_3\rvert + \cdots + \lvert a_{n-1} - a_n\rvert.$$ |