Mexico National Olympiad 2020

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by parmenides51, plagueis

– Day 1

1 A set of five different positive integers is called *virtual* if the greatest common divisor of any three of its elements is greater than 1, but the greatest common divisor of any four of its elements is equal to 1. Prove that, in any virtual set, the product of its elements has at least 2020 distinct positive divisors.

Proposed by Víctor Almendra

2 Let ABC be a triangle with incenter I. The line BI meets AC at D. Let P be a point on CI such that $DI = DP$ $(P \neq I)$, E the second intersection point of segment BC with the circumcircle of ABD and Q the second intersection point of line EP with the circumcircle of AEC. Prove that $\angle PDQ = 90^\circ$.

Proposed by Ariel García

3 Let $n \geq 3$ be an integer. Two players, Ana and Beto, play the following game. Ana tags the vertices of a regular n- gon with the numbers from 1 to n, in any order she wants. Every vertex must be tagged with a different number. Then, we place a turkey in each of the n vertices.
These turkeys are trained for the following. If Beto whistles, each turkey moves to the adjacent vertex with greater tag. If Beto claps, each turkey moves to the adjacent vertex with lower tag.
Beto wins if, after some number of whistles and claps, he gets to move all the turkeys to the same vertex. Ana wins if she can tag the vertices so that Beto can't do this. For each $n \geq 3$, determine which player has a winning strategy.

Proposed by Victor and Isaías de la Fuente

– Day 2

4 Let $n \geq 3$ be an integer. In a game there are n boxes in a circular array. At the beginning, each box contains an object which can be rock, paper or scissors, in such a way that there are no two adjacent boxes with the same object, and each object appears in at least one box.

Same as in the game, rock beats scissors, scissors beat paper, and paper beats rock.

The game consists on moving objects from one box to another according to the following rule:

Two adjacent boxes and one object from each one are chosen in such a way that these are different, and we move the loser object to the box containing the winner object. For example, if we picked rock from box A and scissors from box B, we move scossors to box A.

Prove that, applying the rule enough times, it is possible to move all the objects to the same box.

Proposed by Victor de la Fuente

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5 A four-element set $\{a, b, c, d\}$ of positive integers is called *good* if there are two of them such that their product is a mutiple of the greatest common divisor of the remaining two. For example, the set $\{2, 4, 6, 8\}$ is good since the greatest common divisor of 2 and 6 is 2, and it divides $4 \times 8 = 32$.

Find the greatest possible value of n, such that any four-element set with elements less than or equal to n is good.

Proposed by Victor and Isaías de la Fuente

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6 Let $n \geq 2$ be a positive integer. Let $x_1, x_2, \ldots, x_n$ be non-zero real numbers satisfying the equation

$$\left(x_1 + \frac{1}{x_2}\right)\left(x_2 + \frac{1}{x_3}\right)\ldots\left(x_n + \frac{1}{x_1}\right) = \left(x_1^2 + \frac{1}{x_2^2}\right)\left(x_2^2 + \frac{1}{x_3^2}\right)\ldots\left(x_n^2 + \frac{1}{x_1^2}\right).$$

Find all possible values of $x_1, x_2, \ldots, x_n$.

Proposed by Victor Domínguez
