China Team Selection Test 2020

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– Additional TST

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1 Let ω be a n -th primitive root of unity. Given complex numbers $a_1, a_2, \cdots, a_n$, and p of them are non-zero. Let

$$b_k = \sum_{i=1}^{n} a_i \omega^{ki}$$

for $k = 1, 2, \cdots, n$. Prove that if $p > 0$, then at least $\frac{n}{p}$ numbers in $b_1, b_2, \cdots, b_n$ are non-zero.

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2 Given an isosceles triangle $\triangle ABC$, $AB = AC$. A line passes through M, the midpoint of BC, and intersects segment AB and ray CA at D and E, respectively. Let F be a point of ME such that $EF = DM$, and K be a point on MD. Let Γ_1 be the circle passes through B, D, K and Γ_2 be the circle passes through C, E, K. Γ_1 and Γ_2 intersect again at $L \neq K$. Let ω_1 and ω_2 be the circumcircle of $\triangle LDE$ and $\triangle LKM$. Prove that, if ω_1 and ω_2 are symmetric wrt L, then BF is perpendicular to BC.

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3 For a non-empty finite set A of positive integers, let lcm(A) denote the least common multiple of elements in A, and let $d(A)$ denote the number of prime factors of lcm(A) (counting multiplicity). Given a finite set S of positive integers, and

$$f_S(x) = \sum_{\emptyset \neq A \subset S} \frac{(-1)^{|A|} x^{d(A)}}{\text{lcm}(A)}.$$

Prove that, if $0 \le x \le 2$, then $-1 \le f_S(x) \le 0$.

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4 Show that the following equation has finitely many solutions (t, A, x, y, z) in positive integers

$$\sqrt{t(1-A^{-2})(1-x^{-2})(1-y^{-2})(1-z^{-2})} = (1+x^{-1})(1+y^{-1})(1+z^{-1})$$

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5 Let $a_1, a_2, \cdots, a_n$ be a permutation of $1, 2, \cdots, n$. Among all possible permutations, find the minimum of

$$\sum_{i=1}^{n} \min\{a_i, 2i-1\}.$$

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6 Given a simple, connected graph with n vertices and m edges. Prove that one can find at least m ways separating the set of vertices into two parts, such that the induced subgraphs on both parts are connected.