ITAMO 2020

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by mathisreal

1 Let ω be a circle and let A, B, C, D, E be five points on ω in this order. Define $F = BC \cap DE$, such that the points F and A are on opposite sides, with regard to the line BE and the line AE is tangent to the circumcircle of the triangle BFE.
a) Prove that the lines AC and DE are parallel
b) Prove that $AE = CD$

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2 Determine all the pairs (a, b) of positive integers, such that all the following three conditions are satisfied:
1- $b > a$ and $b - a$ is a prime number
2- The last digit of the number $a + b$ is 3
3- The number ab is a square of an integer.

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3 Let $a_1, a_2, \dots, a_{2020}$ and $b_1, b_2, \dots, b_{2020}$ be real numbers(not necessarily distinct). Suppose that the set of positive integers n for which the following equation: $|a_1|x - b_1| + a_2|x - b_2| + \cdots + a_{2020}|x - b_{2020}|| = n$ (1) has exactly two real solutions, is a finite set. Prove that the set of positive integers n for which the equation (1) has at least one real solution, is also a finite set.

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4 Let ABC be an acute-angled triangle with $AB = AC$, let D be the foot of perpendicular, of the point C, to the line AB and the point M is the midpoint of AC. Finally, the point E is the second intersection of the line BC and the circumcircle of $\triangle CDM$. Prove that the lines AE, BM and CD are concurrents if and only if $CE = CM$.

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5 Le S be the set of positive integers greater than or equal to 2. A function $f : S \to S$ is italian if f satifies all the following three conditions:
1) f is surjective
2) f is increasing in the prime numbers(that is, if $p_1 < p_2$ are prime numbers, then $f(p_1) < f(p_2)$)
3) For every $n \in S$ the number $f(n)$ is the product of $f(p)$, where p varies among all the primes which divide n (For instance, $f(360) = f(2^3 \cdot 3^2 \cdot 5) = f(2) \cdot f(3) \cdot f(5)$).
Determine the maximum and the minimum possible value of $f(2020)$, when f varies among all italian functions.

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6 In each cell of a table 8×8 lives a knight or a liar. By the tradition, the knights always say the truth and the liars always lie. All the inhabitants of the table say the following statement "The number of liars in my column is (strictly) greater than the number of liars in my row". Determine how many possible configurations are compatible with the statement.

