Greece Team Selection Test 2020

www.artofproblemsolving.com/community/c1594871

by parmenides51, tastymath75025, naman12

1 Let $R_+ = (0, +\infty)$. Find all functions $f : R_+ \to R_+$ such that

$f(xf(y)) + f(yf(z)) + f(zf(x)) = xy + yz + zx$, for all $x, y, z \in R_+$.

by Athanasios Kontogeorgis (aka socrates)

---

2 Given a triangle ABC inscribed in circle $c(O, R)$ (with center O and radius R) with $AB < AC < BC$ and let BD be a diameter of the circle c. The perpendicular bisector of BD intersects line AC at point M and line AB at point N. Line ND intersects the circle c at point T. Let S be the second intersection point of cicumcircles c_1 of triangle OCM, and c_2 of triangle OAD. Prove that lines AD, CT and OS pass through the same point.

---

3 The infinite sequence $a_0, a_1, a_2, \dots$ of (not necessarily distinct) integers has the following properties: $0 \le a_i \le i$ for all integers $i \ge 0$, and

$$\binom{k}{a_0} + \binom{k}{a_1} + \cdots + \binom{k}{a_k} = 2^k$$

for all integers $k \ge 0$. Prove that all integers $N \ge 0$ occur in the sequence (that is, for all $N \ge 0$, there exists $i \ge 0$ with $a_i = N$).

---

4 Let a and b be two positive integers. Prove that the integer

$$a^2 + \left\lceil \frac{4a^2}{b} \right\rceil$$

is not a square. (Here $\lceil z \rceil$ denotes the least integer greater than or equal to z.)

Russia
