

USA Team Selection Test for IMO 2021www.artofproblemsolving.com/community/c1601449

by v_Enhance

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- 1** Determine all integers $s \geq 4$ for which there exist positive integers a, b, c, d such that $s = a + b + c + d$ and s divides $abc + abd + acd + bcd$.

Proposed by Ankan Bhattacharya and Michael Ren

- 2** Points A, V_1, V_2, B, U_2, U_1 lie fixed on a circle Γ , in that order, and such that $BU_2 > AU_1 > BV_2 > AV_1$.

Let X be a variable point on the arc V_1V_2 of Γ not containing A or B . Line XA meets line U_1V_1 at C , while line XB meets line U_2V_2 at D . Let O and ρ denote the circumcenter and circumradius of $\triangle XCD$, respectively.

Prove there exists a fixed point K and a real number c , independent of X , for which $OK^2 - \rho^2 = c$ always holds regardless of the choice of X .

Proposed by Andrew Gu and Frank Han

- 3** Find all functions $f: \mathbb{R} \rightarrow \mathbb{R}$ that satisfy the inequality

$$f(y) - \left(\frac{z-y}{z-x} f(x) + \frac{y-x}{z-x} f(z) \right) \leq f\left(\frac{x+z}{2}\right) - \frac{f(x) + f(z)}{2}$$

for all real numbers $x < y < z$.

Proposed by Gabriel Carroll
